DOCUMENT RESUME

ED 096 179	SE 018 479

TITLE	Multi-Sensory Aids in the Teaching of Mathematics. National Council of Teachers of Mathematics, Yearbook 18 [1945].
INSTITUTION	National Council of Teachers of Mathematics, Inc., Washington, D.C.
PUB DATE	45
NOTE	470p.
AVAILABLE FROM	National Council of Teachers of Mathematics, Inc., 1906 Association Drive, Reston, Virginia
EDRS PRICE	MF-$0.75 HC Not Available from EDRS. PLUS POSTAGE
DESCRIPTORS	*Audiovisual Aids; Exhibits; Instructional Films; *Manipulative Materials; Mathematical Models; *Mathematics Education; *Multimedia Instruction; Multisensory Learning; Reference Books; Slides; Tape Recorders; Visual Aids; *Yearbooks

ABSTRACT

The chapters in this yearbook describe some of the experiences the individual authors have had in making and/or using multisensory aids in mathematics instruction. A number of different aids are noted to indicate the range of possibilities. Three more chapters set the stage for the remainder of the book which is divided into the following sections: drawing and design; demonstrations and exhibits; models and devices; instruments and tools; materials for construction of models and devices; and slides, films, three-dimensional projection, and equipment. In the appendix a short description of individual models and devices is given by content areas. (LS)

ED 096179

The National Council of Teachers of Mathematics

EIGHTEENTH YEARBOOK

Multi-Sensory Aids in the Teaching of Mathematics

COMPILED BY

THE COMMITTEE ON MULTI-SENSORY AIDS
OF THE NATIONAL COUNCIL
OF TEACHERS OF MATHEMATICS

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BUREAU OF PUBLICATIONS
TEACHERS COLLEGE, COLUMBIA UNIVERSITY
NEW YORK, 1945

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National Council of Teachers of Mathematics
1201 Sixteenth Street, N.W.
Washington 6, D. C.

Second Printing, April, 1946
Third Printing, September, 1947
Fourth Printing, August, 1958

Manufactured in the United States of America

EDITOR'S PREFACE

This is the eighteenth of the series of Yearbooks started in 1926 by The National Council of Teachers of Mathematics. The titles of the preceding Yearbooks are as follows:

1. A Survey of Progress in the Past Twenty-Five Years.
2. Curriculum Problems in Teaching Mathematics.
3. Selected Topics in the Teaching of Mathematics.
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5. The Teaching of Geometry.
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12. Approximate Computation.
13. The Nature of Proof.
14. The Training of Mathematics Teachers.
15. The Place of Mathematics in Secondary Education.
16. Arithmetic in General Education.
17. A Source Book of Mathematical Applications.

It is unfortunate that this Yearbook is late in appearing, but circumstances over which we have had no control made it impossible to get the Yearbook out on time. We hope that the quality of the material and the nature of the articles will make up for the lateness in the appearance of the volume.

Because of possible changes in price in these uncertain times, all prices of materials referred to in this book should be carefully verified before such materials are ordered.

We do not claim that this Yearbook contains all the possible teaching aids that may have been suggested, but we hope that many teachers will find it the source of great help and inspiration.

As Editor I wish to express my personal appreciation to Professor Hildebrandt and his committee, who have done the main work for this volume, and to The National Council of Teachers of Mathematics for their continued support and advice.

W. D. Reeve

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5. Raise the general level of instruction in mathematics.

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INTRODUCTION

TEACHING AIDS in mathematics are not new. The last hundred years have brought us the telephone, the phonograph, the radio, television, the silent and sound motion picture, the stereoscope, the three-dimensional colored pictures on lenticulated film and the Polaroid three-dimensional pictures, and motion pictures in color. These inventions and developments are being used in many forms in our schools at the present time. It is only natural that mathematics teachers, too, consider the possible adaptation of these materials to the improvement of instruction in their field.

The eighteenth annual convention of the National Council of Teachers of Mathematics held in Chicago on February 19, 1937 revealed a number of the uses to which mathematics teachers are putting these recent developments. For example, its honorary president, Professor H. E. Slaught of the University of Chicago, could not be present at the dinner; but his greeting[1] to the members had been recorded and was reproduced over the loud-speaker system of the banquet room. Some day we may have more recordings from others who have contributed so much to the teaching and the development of mathematics. One of the groups at the discussion luncheon, under the leadership of Miss Mary A. Potter, prepared a list of materials on the topic "The Kinds of Pictures We Use in Teaching Mathematics" and this list was made available later in mimeographed form. At this convention, recent silent and sound films closely related to mathematics were shown, and a large exhibit of charts, graphs, models, and other materials prepared by teachers in the Chicago schools indicated further worthwhile aids. It was apparent that mathematics teaching is being carried on with various kinds of materials: that we learn by seeing and hearing as well as by doing.

The Visual Aids Committee of the National Council of

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[1] Slaught, H. E., "Greetings to My Friends of the National Council," *Mathematics Teacher*, Vol. XXX, pp. 186-187, April, 1937.

Teachers of Mathematics, under the chairmanship of Professor E. R. Breslich, was later appointed and this committee publicized further suggestions and improvements. As the field continued to grow, the suggestion was made that a report be prepared on the work in multi-sensory aids carried on by mathematics teachers throughout the country. The Board of Directors of the National Council, at its annual convention in Atlantic City, N. J., in 1941, voted that the report be prepared as a Yearbook of the Council.

The present report does not claim to be an exhaustive study of all the aids which mathematics teachers have used and can use to good advantage. The individual articles describe some of the experiences which their authors have met and indicate as large a number of different aids as possible. It is hoped that the report will be followed in a few years by one showing improvements and changes which have kept pace with the progress of such aids in the world about us.

Many of these articles are the results of papers and studies reported at conventions of the National Council held at Baton Rouge, Atlantic City, Boston, Bethlehem, San Francisco, and Denver. Discussions devoted to multi-sensory aids were held at most of these meetings; they have been the impetus for many new suggestions and experiments.

It is not possible to indicate all the members and friends of the National Council who have contributed materials, references, and suggestions. A list is given at the close of this Introduction. To them and others whose names may have been omitted the Committee wishes to express its deepest gratitude for the help given. It is most gratifying to note that when mathematics teachers pool their resources, much can be accomplished. If this Yearbook were inscribed with a dedication, it would be to those teachers who have done and are doing constructive work for the improvement of mathematics teaching.

It is fitting that we express our appreciation to one teacher in particular: Miss Mary A. Potter, president of the National Council from 1940 to 1942. During her years in office, Miss Potter rendered many services to this Committee and arranged programs on multi-sensory aids at the several conventions of the Council. In addition to her many duties and responsibilities as president, she gave unhesitatingly of her time, advice, and assistance to problems relating to the development of this Yearbook.

We wish also to voice our thanks to every member of the Board of Directors, who, during the years of the preparation of this Yearbook, have recommended the preparation and publication of this report. To Professor W. D. Reeve, with the enviable record of seventeen yearbooks for the National Council to his credit, any words of appreciation are inadequate for the interest, help, and cooperation he has so generously given.

E. H. C. H.

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Multi-Sensory Aids
in the
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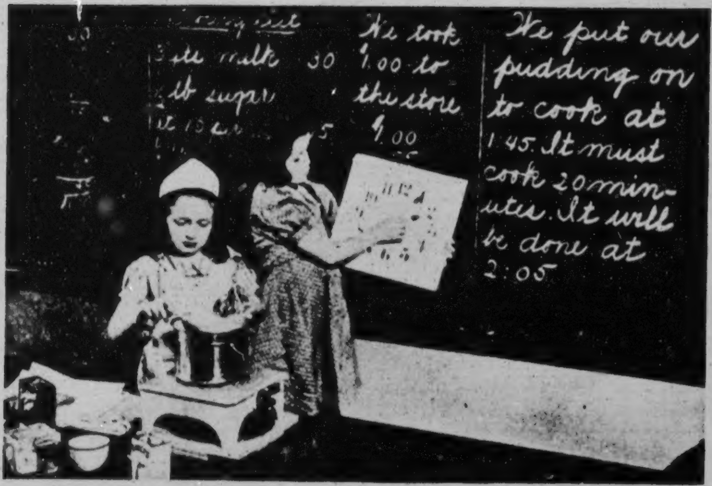
Arithmetic, measurement, and even graphs are to be found in the growing beans and potatoes in the second grade classroom.


We took
$1.00 to
the store
$1.00
We put our
pudding on
to cook at
1:45. It must
cook 20 minutes. It will
be done at
2:05

There is plenty of arithmetic in the cooking lessons that come to the third grade when the "kitchenette" is sent to a school—buying the supplies, checking the change, measuring the ingredients, timing the pudding, making paper doilies, setting the table, serving the class.

MULTI-SENSORY AIDS: SOME THEORY AND A FEW PRACTICES

Edith Sifton

MULTI-SENSORY AIDS? A new term? What does it mean? New, yes, but the questions can be answered very easily by recalling some of the educational history of the past few years. Remember those first projectors and their glass slides, and how progressive some of us felt when we began to use them in our classrooms? Shortly, we brought movies to our students and began to talk to each other about the various kinds of "visual aids." Next, we added ear appeal, and "talkies" became, in our language, "audio-visual aids." Now, we find ourselves—teachers of mathematics—stepping to the fore with an entire yearbook devoted to "multi-sensory aids." We are noting that children learn through other avenues than their eyes and ears—for example, their hands!

But, in taking such a step, it seems that we have moved somewhat ahead of the procession. Search as we will, we cannot find, to date at least, the word "multi-sensory" listed in any of our numerous books dealing with the theories of education.

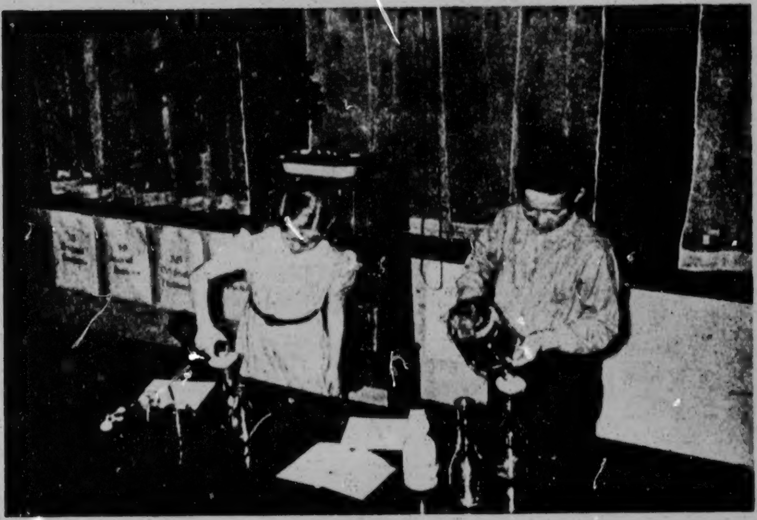
As teachers of mathematics, however, we should have no difficulty in finding our way in this new field of educational thought, for it dovetails so neatly with today's theories in our own subject area. We know, in the first place, that of all the subjects taught by teachers and studied by children, mathematics is, by its very nature, one which gains greatly by the use of multiple and simultaneous impacts upon the mind of the learner. We know, too, that for *our* pupils, learning *must* be something more than seeing and hearing; for them mathematics *must* be a means for *doing* things in the classroom as well as in a later, workaday world.

Under such circumstances, we need hardly be surprised to find that many of today's teachers of mathematics have devised and used a goodly number of multi-sensory aids, even though educational philosophers have neglected to develop any theory to guide those teachers who are in the front lines. We have, in fact, but to look about us to find incidents and examples that will give us data for evolving some theories of our own, and for listing some very effective practices, in this new field.

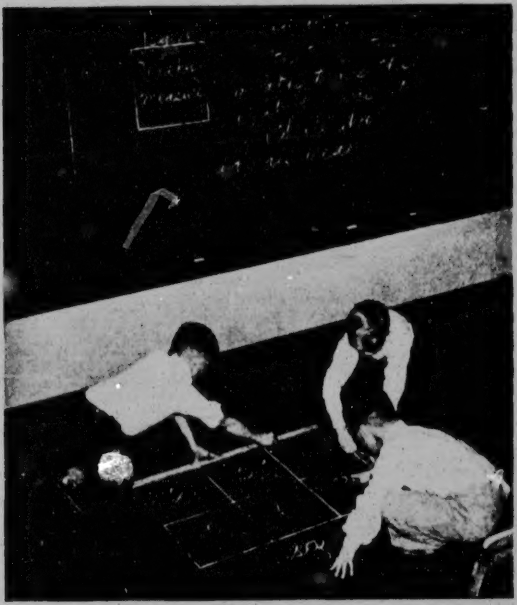
To begin, consider an incident that occurred recently at the University of Washington during a meeting of some of the members of the Pacific Northwest Resources Workshop. In the course of the informal discussion one of the group "wondered," aloud, just what any mathematics teacher would find useful in a study of resources. The staff member in charge of the mathematics section of the Workshop met the challenge promptly. Her first words were: "Well, if students are going to 'mathematick', they will have to be given something to 'mathematick' about."

If we grant mathematics should be taught as though the word were a transitive, active verb, we may have a clue to one of the reasons why we have found it rather difficult to devise, or obtain, visual aids that completely satisfy us. Do not misunderstand—we need visual aids. We cannot ignore them, nor discount the fact that the impression a picture makes on the mind of a child is often much more lasting and vivid than one made by words, printed or spoken. Visually, we can give our pupils understandings, meanings, and appreciations that would otherwise be impossible. There are times, also, when pictures can do a very good job of explaining difficult points. Consider, for example, those children who, with such seeming trust, ask us to "explain the theory of relativity." We can satisfy such requests by showing them the film on Einstein's theory. The idea is presented upon a level at which children feel at home, and they can see the picture several times, with increasing understanding each time. Again, we need not more than one showing of such films as "The Vernier" and "The Micrometer" to find ground for the belief that the present crisis in training industrial workers will, in the end, bring us other good films in the field of mathematics.

On the whole, it certainly is not without reason that the expression "I see" so often means "I understand." There must also be very real reasons why a child instinctively wants to touch everything he sees; why he wants to learn actively rather than passively; why he takes his toys apart to see what makes them go;



Real cups, pints, quarts, and gallons mean in grade four real problems in multiplication, division, and fractions.



A square yard, drawn on the floor and renewed from day to day, walked over and looked at many times, is more than another abstraction for the fifth grade.

why, in high school, his hands fairly itch to get hold of a transit, while a picture of one arouses only casual interest.

Possibly just a little theorizing is in order in connection with the responses of teachers and parents to this youthful instinct to learn by handling, manipulating, and "doing." Looking about us we find, apparently, at least three schools of thought in the field. We might agree with the first up to a certain point—that a child may need to have his hands slapped until he gets over that impulse to pull the coffee-maker over onto himself. If this technique were repeated too often, however, there would be some chance of slapping curiosity right out of the child's mental make-up. Teachers, at least, have found intellectual curiosity too valuable an incentive to learning to want it eliminated, or even so much as dulled.

Our second school of thought seems to believe that whatever attracts the child's eyes, and therefore his hands, should be placed upon the highest shelves, or be kept out of sight entirely, back of closed doors. The bric-a-brac is too valuable to be entrusted to the mercies of awkward fingers. There may be a possibility that mathematics teachers feel that their particular set of bric-a-brac is too complicated for unskilled hands. Followed consistently, this second method of dealing with a child's attempts to explore his world might conceivably lead to a fairly typical case of intellectual malnutrition.

Scrutinizing the third school of thought, we find people who see to it that children have supervised, not to say planned, opportunities to handle anything from which they can learn, if they but show interest in it. Some even go to the length of stimulating curiosity and interest in seeing, handling, and manipulating objects. Certain items among our mathematical bric-a-brac may have to be simplified a bit, but many of them can be given to our children to handle and "work," with benefit to all concerned.

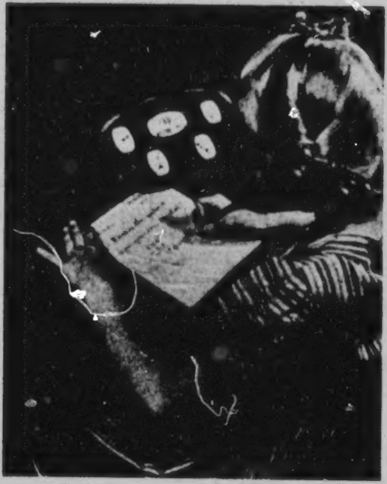
By the way, there is more back of the attitude of this last group than a laudable desire to see children learning happily. Put something concrete into the hands of a child, something that will enable him to enter actively into the learning situation, and auditory, visual, oral, tactual, and muscular sensations unite in a drive that has real power in forming new thought patterns.

All this theorizing has its place, no doubt, but in the end, putting theory into active practice is the only way to bring in dividends. If we turn to the work of some of the mathematics teachers belonging to that third school of thought, we can see how they do it, and get, as well, a glimpse of the value of the dividends. From each of them we can take away ideas that can be applied to our own work, no matter what its level—in the end many more ideas than could be gathered from whole volumes of expositions on theory. So let us visit a few of those teachers who have been making the word "mathematics" a transitive, active verb for their students.

As you begin these classroom visits imagine that you are a pupil in the second grade. The arithmetic course of study for this grade declares that you should be taught to tell time by hours and to five-minute subdivisions. You might be shown the clock on the wall, and have carefully explained to you just how you are to read the time. Your teacher might also use that cardboard clock dial, setting the hands to show the time for recess or for lunch. She may even ask *you* to set it for the time to come to school, or the time to go to bed. But suppose your teacher also lets you make a clock dial of your own, with movable hands; and not only has you set it, for the hours and minutes you *must* learn, but lets you set it just for fun, whenever you have finished your work ahead of the rest of the class. Suppose she is always willing to pause for a second or two to answer such a question as, "Does my clock say quarter past three?" Which method would you, a seven-year-old, choose?

And next, suppose you have moved on to the third grade and are in the midst of your first hand-to-hand struggle with remainders in division. You are dividing 11 by 3. Miss Smith could go over the work carefully at the board, could make matters quite clear to you with assorted additions, subtractions, examples, and problems. She might also make a row of eleven marks on the board, cross them off three at a time, and prove to you that there is a remainder of two. But suppose that, instead, she tells you to take some of the small log-cabin blocks, some of the seeds left from the Hallowe'en pumpkin, or some of those little clay balls you are using for strawberries in the play-store. She asks *you* to

An old speedometer, a child, and decimal fractions come together in a "learning situation" in the sixth grade.



What is pi? It can be more than rote memory of a certain combination of digits, decimal point, and formula for these seventh grade pupils.

show *her* how many threes can be made with eleven of them. You try them in little rows, like soldiers on parade; you put them in little clover-leaf groups; each time you have two left over. Wouldn't *you* prefer to acquire division facts with your hands and eyes, as well as with your head? Wouldn't you understand them better, and remember them longer?

You will find that elementary, intermediate, and upper grade teachers are also giving their pupils understanding of the various phases of mathematics by appealing to the mind through the hands as well as through the eyes and ears. In a fifth grade classroom we find a group of children deeply engrossed in using an abacus. They are adding numbers so accurately and rapidly that we have difficulty in following the process. Units, tens, hundreds, thousands have new and concrete meanings for these boys and girls. Across the hall in the seventh grade arithmetic class some boys are increasing their understanding of the tens system by putting the machinery of a dismantled light meter into action. As we look over the shoulders of some other pupils in the sixth grade, we find them operating an old speedometer, setting it for some original problems in decimals.

Going on to the case of fractions, we find that a six-year-old needs no more formal instruction in the meaning of one-half than to have his teacher say to him, as he divides that apple for the birthday party, "If the apple isn't cut in *half,* we always give our visitor the larger part." If, a few years later in his career, the child forgets that 3/6 is the equivalent of 1/2, or that 2/6 equals 1/3, he will be handed those brightly colored paper plates cut into halves, quarters, thirds, sixths, and eighths. As he fits the red plate, cut into sixths, into the yellow one, divided into halves, or into the green one with its thirds, active hands as well as active mind will help him learn the "lesson."

In the ninth grade general mathematics class, boys and girls are reviewing decimal fractions. The instructor is discussing precision measurements. That enlarged picture of a micrometer caliper on the bulletin board is interesting to the class and, with some explanation, moderately comprehensible. The pupils pay closer attention, however, when the teacher uses an actual micrometer and measures the thickness of a dollar bill right before their eyes. But the whole thing begins to "click" with them when he gives them each a turn at the instrument, and hands them a leaf-feeler gage as well. They finger and measure these pieces of metal. They turn the thimble to discover, if they can, any visible change in the opening as the spindle moves through five one-thousandths of an inch. Playing? Possibly. But gaining also some appreciation of the kind of measurements made by whole armies of today's workmen and, in addition, at least an inkling of the meaning of approximation.

When it comes to acquiring some feeling for the "size" of large numbers, one must have more than a good imagination or a chance to gaze at a star-sprinkled sky. Here again teachers with ingenuity have brought the hands of their children to bear upon the task.

Take the case of the fifth grade that was learning to read and write numbers in the millions. Had any of them ever seen a million of anything? One boy offered the suggestion that there were millions of tiny pebbles in that fine gravel that covered the playground. How much would it take to make a million pebbles? The children thought a gallon—and could they count it and see? They compromised by counting a pint to start with. Each counted his share of the pint, and the amounts were added. The class decided it would really be better to finish by calculation; so they multiplied to find the number of pebbles in a gallon. Finally, by rounding off and dividing, they learned that they would have had to bring 125 gallons of gravel into the classroom in order to have one million pebbles.

Again, a seventh grade, trying to compare government finances with their own, devised a way of getting an idea of the size of a billion dollars. Using some small kindergarten blocks in lieu of dollars, they took turns at counting the blocks for one minute at a time. The average number counted per minute was determined, and then, by dividing, the class found it would take the average pupil almost seventy-five years to count a billion dollars. To those boys and girls a billion now means more than a certain number of zeros trailing after the digit 1.

In teaching the various topics and units in measurement encountered recurrently in the course of study, many teachers bring the tables concerning pints and quarts, square feet and square yards right out of the textbook and into the children's own world. In the second grade we see small boys and girls discovering for themselves some of the facts about liquid measure by pouring water into a large assortment of quart, pint, and half-pint containers they have found in their homes. A little later in the year the teacher takes advantage of a hot day to increase their knowledge of measurement to a fraction of an inch by allowing each child to measure and fold a paper drinking cup of his own. Just before Christmas, a fifth grade arithmetic teacher and an art teacher collaborate in having the children make gift boxes. In the arithmetic class, drawing the plans calls for estimation, measurement, addition and subtraction of fractions.

There are pupils in the seventh grade who persist, in spite of teacher demonstration and exhortation, in working their area problems on the basis of twelve square inches to the square foot. However, after drawing a square foot on wrapping paper, cutting a square inch out of cardboard, and filling that larger square with smaller ones by tracing around the cardboard, the children *know* just why they must use 144, and not 12, in their problems.

On making their first bow to areas, the children in one fifth grade are allowed to draw a square yard upon the floor, and to divide it into square feet. Renewed from day to day, walked over, as well as looked at, "square yard" becomes more than just another abstract word. In fact, the children can almost feel, as well as see, square yards in Mother's new linoleum and in the garden plots they are planning for their science project.

If the measurement we teach is to be practical, children must be able to estimate size and measure objects at least as effectively as they work book problems listed under the topic. As one case in point, consider another class in the fifth grade. The question of the size of Captain Vancouver's boat came up in the course of a discussion. One pupil thought it must have been as long as the school grounds; another that it could have been put into the hall. There was nothing to do but to measure hall and playground. The second pupil won the argument, but, with little more than a hint from the teacher, the class was off on a project of estimation and measurement that took in numerous and rather astonishing items which the children found interesting. Included were the size of the "Saratoga," heights of school desks, lengths of the curtains at home, the distance to the University bridge, the length of each child's stride, the number of strides in a quarter mile. Need one add that the tables of measurement customarily presented at this level were also involved in the many problems posed by students as well as by the textbook? Or that there were opportunities also for some telling demonstrations of the advisability of using good judgment by suiting the size of a unit to the size of the quantity to be measured?

Moving farther along our educational assembly line, we find that multi-sensory methods render equally effective assistance in overcoming some of the difficulties of high school pupils and their teachers. A plane geometry class is learning the theorems dealing with the measurement of angles in a circle—hardly active mathematics. Suppose, however, the students use homemade sextants—perhaps of their own manufacture—and try their hands at the business of shooting the sun and determining the latitude of their city. They may not, as a result, be able to navigate the Pacific, but for them the measurement of central angles comes to life. Again, steel tape, shadows, and mirrors can be used to create problems that not even a sophomore can consider academic. Give a boy a ruler and a calling card and he can construct a small model of a cross-staff that enables him to make and solve some problems of his own in proportion, and that introduces him to the principle of the instrument with which, as a young navy officer, he may some day keep his "battlewagon" in formation. A larger piece of cardboard, a string, and some sort of plumb bob can be turned into an "altimeter" that, having been used to measure inaccessible distances, leads to an intelligent use of the tangent ratio. More cardboard, cut into strips, can be made into reasonably workable pantographs, into instruments for trisecting certain angles, or into parallel rulers. Brass paper fasteners will do for joints.

The units on loci are often made the most abstract and the most disliked in plane geometry. Some of the more analytical minds of a class will find pleasure in using compass and ruler to discover for themselves new loci, such as conic sections. But for students who are of a more concrete and practical turn, using a

The tools by which a workman applies the mathematics of the circle can also be the tools his son uses for this problem in the ninth grade.

Why should geometry students be left in ignorance of the existence, use, and geometry of the pantograph? (Tenth grade) couple of student-made models of the Osborne fire-finder, some maps furnished by the supervisor of the near-by national forest reserve, and locating, thereby, some hypothetical forest fires will give a lift that makes lighter work of the whole topic. One such experience, and boys and girls will observe and report many of the innumerable applications of loci to be found all about them.

Going on to solid geometry, the fact that volumes of similar solids vary as the cube of corresponding dimensions is merely book-learning to most boys and girls. Bring out, however, two tea boxes, approximately cubes, without disclosing the fact that they formerly held a half-pound and a pound, respectively. Let the students first estimate the ratio of the volumes, and then measure the dimensions and determine the ratio by calculation. The astonishment on the faces about the classroom is evidence enough that some active thinking is being done.

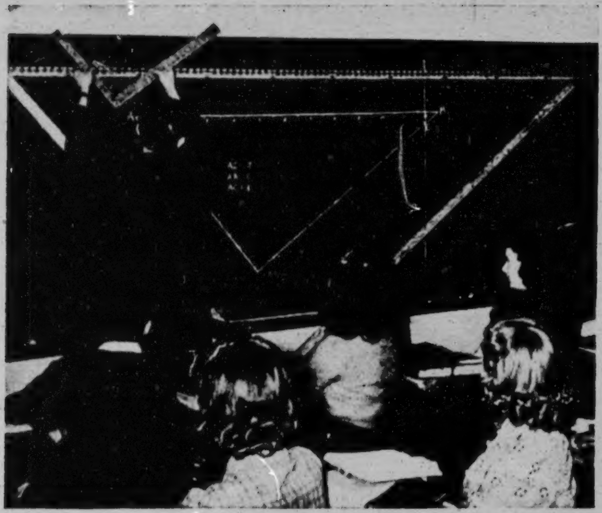
It took two classes in solid geometry to make one teacher realize the effectiveness of letting students use their own hands. Some of the students of the first class were having the usual difficulty in visualizing the solids generated by rotating plane figures. One of the boys watched the instructor whirl some cardboard forms and noted, evidently, the indifference of the results. A few days later he came to class with a set of triangles and rectangles that he had made by bending wire. He had made each with a sort of stem, and had also made a stand from one of his mother's shortening cans by punching holes in the top and bottom. Winding the stems with string, and inserting the stems in the holes in the stand, he whirled the figures like tops, and so formed ghostly solids. The succeeding semester that set of figures was put on a table a few days before the rotation problems were to come up. Students of this second class played with the figures before and after class sessions, with what seemed to be almost idle pleasure. When it came to assigning the problems in this class, however, there was no demonstrating for the teacher to do—the students' visualizations were all nicely taken care of!

Such instances are all very well, you may say, but after all, mathematics means solving problems. Agreed! And, one might add, it is one thing to solve even the most practical of problems that have been nicely corralled in a textbook, and quite another to discover some for yourself, right on the open range, in work or in play, in home or in community.

Here again, teachers in every grade can show us just how well children will respond to discovering and solving original problems. One second grade wrote a whole series of problems centered about the aquarium they were setting up. How much would it cost to buy the fish, their food, and the plants? How much more sand would they have to put in if it was to be two inches deep? How many quarts of water would they have to pour in if one gallon was needed for each inch of fish? Further details are hardly needed to show that such a project can involve a number of multisensory appeals and several learnings, along with the mechanics of addition and subtraction.

Of course such programs for developing original problems become more complicated in later work; but consider one example from the sixth grade. In the textbook, there was a unit in which the problems were based on transportation figures. First the class made some original additions to that list, based upon similar facts observed on summer trips. But before long their study led them right out of the textbook and transportation, and into a discussion of the post office. Interest was so keen that the teacher arranged for a conducted tour of the terminal station, with each pupil responsible for bringing back at least one interesting number fact. Back in class, the number facts were compiled into a single list, and from it the children wrote an entirely original set of problems. One problem, for instance, called for finding the cost of sending a cake by air mail to a brother in camp; another, for the number of letters passing through the canceling machines in a day; a third, for the average number of people served by each local carrier. In the end, the pupils' own set of problems was longer than any which a textbook would have provided. Furthermore, there could hardly have been a better way of learning to solve problems than by discovering, analyzing, and then formulating some for themselves.

One might go on indefinitely with examples of the ways in which various teachers are coordinating children's eyes, ears, and hands, in giving them opportunities to learn how to "mathematick." Having once seen the response that children of all ages



Try it for yourself—a "rafter" and a carpenter's square can be made to add to the understanding of the "theory" back of some practical geometry. (Tenth grade)

You may be able to buy, and even use, a slide rule without understanding it, but you cannot make one and fail to learn something of the principles it employs. (Twelfth grade) make to such methods, most of us could never again plead lack of class time as a reason for not using them. One may have to admit that planning multi-sensory activities takes ingenuity and a dash of imagination besides; that discovering openings for such planning takes sensitiveness to the signs of dawning child interest, and, above all, an appreciation of the pervasiveness of mathematics. But handling such projects in class consumes, generally, not more than ten or fifteen per cent of the mathematics period. During that time, teachers not only develop some real understandings and appreciations, but build up, as well, a "head of steam" that carries pupils through whole days of purposeful work on exercises and problems.

A MATHEMATICS CLASSROOM BECOMES A LABORATORY

L. Grace Carroll

A CENTURY AGO such philosophers as Huxley, Spencer, and Rousseau felt and preached the need for functional education. As a result, they succeeded in introducing science and the scientific method into secondary education; the concepts of learning by doing and experimenting were concomitants. Curriculum offerings introduced since their time have, like science, been taught in laboratories and workshops by means of the laboratory method.

But there have been various reasons why mathematics teaching has adhered in most respects to the old method of question and answer: tradition, the regime of conservatism, lack of appreciation and understanding on the part of teachers, lack of realization of the needs for and the opportunities offered by a laboratory set-up, indifference, poor organization, unwillingness to try anything new, possible difficulties in administration, expense, and lack of ingenuity.

Multi-sensory aids are not new. They were introduced and recommended by Rousseau. Later the philosophy of learning by doing was popularized in the elementary school by John Dewey and his followers in the activity program. Today every elementary school teacher who considers herself even pseudo-progressive aims to make her classroom a laboratory. Consequently, the modern elementary school classroom is a workshop in which children do things and grow in the process.

But, in the high school, classical subjects which antedate Huxley, Spencer, Rousseau, and the activity program are all too commonly taught by the medieval methods of lecture, question, and answer. Except, perhaps, for the differences in dress and attitude of the students, a casual visitor might be unable to tell the difference between many 1944 classrooms in mathematics and their prototypes of the Middle Ages.

The first requirement for a mathematics laboratory is the will to start. The chief need is the initial effort to organize available materials, thus providing opportunity to work with these materials and to discover mathematical laws and concepts in an objective way. With some help and direction from the teacher, considerable equipment will accumulate as the work proceeds. This pupil-inspired material is often more valuable than commercial equipment. It tends to stimulate interest, to create desirable attitudes, and to furnish the thrill that comes from accomplishment.

This section of the Yearbook uses the New Rochelle Public Schools as an illustration of work being done in a modest effort to improve present practices in the teaching of secondary school mathematics. It does not attempt to describe an ideal situation, first, because we do not claim to have an ideal situation, and second, because there is no end to progress. Every end is but a beginning and "under every deep a lower deep opens."

The present program had its beginnings in discussions among groups of mathematics teachers, some of whom felt that improvements were needed in mathematics, both in subject matter and in method. The Superintendent thereupon appointed a committee to study the status of mathematics in New Rochelle and to make recommendations for change, if change were needed. Committee members were released from regular classroom duties to carry through this study. They observed classes, interviewed teachers, conferred with students, and summarized their findings. Their report included detailed information on present practices and a summary of suggestions for improving them.

Through the exchange of ideas and through the combined efforts of administration and committee, New Rochelle has effected many constructive developments. One of these was an effort to improve the teaching of mathematics through the use of visual aids. A mathematics exhibit to which both elementary and secondary schools contributed was displayed in the museum at the Administration Building. It included, among other things, old mathematics textbooks and other books of historical value; mathematical instruments, both commercial and pupil-made; an antique quilt of "mariner's compass" pattern; lantern slides; stereographs; film strips, motion-picture films; solid geometry models made of cardboard, cellophane, and wood; charts; drawings; photographs of pupils' activities; and other articles of mathematical interest.

Subsequently two mathematics laboratories were started experimentally. As the laboratories began to develop, questions were raised by teachers and pupils. Among them were such questions as these:

What is a mathematics laboratory? How will a mathematics laboratory differ from any other classroom?

Can more than one teacher use the same mathematics laboratory?

Will a classroom laboratory contribute to a program of directed study?

Can mathematics be taught as a laboratory subject in the same way as the arts and other sciences?

What extra expense will be involved in developing a laboratory?

What equipment will be needed?

Why are our old methods unsatisfactory? We have no fault to find with the status quo.

This article will describe how some of these questions were answered in our own situation.

What is a mathematics laboratory? It is a place for learning by doing. Any laboratory is a workshop. One concept of a mathematics laboratory is an environment in which a pupil learns efficiently and more meaningfully the mathematics he should learn anyway. The teaching is informal and individual, with all possible opportunities offered to discover or rediscover mathematical truths and laws. Visual aids are used in making the work as objective as the problems require and the facilities permit. Just how this can be done will vary, but a few illustrations of ways in which we have done it are given in this article.

How will a mathematics laboratory differ from any other classroom? The answer to this question is found in part in any classroom used by a progressive teacher. Equipment (or its lack) is not the only distinguishing factor. There is an informal type of teaching and a type of directed study, both of which vitalize the teaching of mathematics. These help to make the room a workshop instead of a lecture room. There is more "If . . . Then" discussion, followed by discovery and increased development of power of discovery. An extended list of materials and equipment suggested for inclusion in a laboratory is given on pages 26-29.

Can more than one teacher use the same mathematics laboratory? This is possible and in some cases it may be very desirable. But, in general, a teacher prefers to convert her own classroom into a laboratory. However, she usually is willing to share materials and ideas with other teachers who are actively interested.

Can mathematics be taught as a laboratory subject in the same way as the arts and other sciences? Perhaps the answer is best expressed by saying that no two laboratory subjects are taught in exactly the same way. Here is an illustration of a topic taught in a very informal way:

The topic was "Straight-line Graphs." Thermometers and carefully constructed diagrams of thermometers were studied; the way the readings changed together was expressed in words and by means of the formula. This same relationship was expressed by means of a table. The students learned to interpret a graph involving signed numbers, and then expressed by means of a graph the way degrees centigrade and degrees Fahrenheit change together. They made the first graphs by locating points. From a study of their work and observation of the formula, they discovered the meanings of slope and intercept. With most of the graph work that followed, they preferred to use the slope-intercept method.

This is a procedure any class could use. There had been so much group interest and careful work that the teacher thought—aloud—that it would be helpful if future classes could see pictures of this class at work and could know how this class made its discoveries. She had made the comment hoping that it would result in a meaningful activity. She was not disappointed. One pupil interested in photography as well as in mathematics suggested that he take pictures of the class at work. The result was approximately sixty 2 x 2 lantern slides showing the development of how numbers change together and the class as they worked discovering functional relationships. The slides showed the initial steps in making a graph and the processes required in reasoning and drawing the graph by the slope-intercept method.

Because of the picture-taking, the early stages of the study were repeated, but with more meaning and added interest. The students were proud of their accomplishment. They were thrilled when the pictures of the class and their discoveries were shown on the screen. In addition to the lantern slides, there was a written account of the project illustrated by pictures mounted and used for a bulletin board exhibit.

The aim had been achieved. The pupils understood the meaning of how numbers change together and could express this functional relationship in a manner that suited their convenience.

Thus these students carried through the experiment. As the project progressed, members of the class who were not working on the experiment at a given time were busy developing other projects along lines of their own interests. However, as occasion arose, they made contributions to the project in hand and so always had discoveries to challenge their interest.

This experiment also helped in furthering experiments with centigrade thermometers which were being carried on as part of the science work of this group of students.

Another experiment was finding the meaning of the formula $V = lwh$. A student having some difficulty with the problem was willing (in fact, eager) to construct a rectangular solid and show by the construction of cubic units and the placing of these units how and why the number of cubic units in the rectangular solid is equivalent to lwh. The rectangular solid and the cubic units were made from heavy cellophane and put together with Scotch tape. The boy had been advised to make one of the little units of measure, but he insisted on making the number required to fill the rectangular solid. Much disturbed when these units did not fit exactly, he wanted to repeat his experiment. This proved an opportune time to study error and sources of error. The discussion and conclusions led to an appreciation of precision in measurements and a desire for accuracy. Not only did this class find the models helpful, but the project was used subsequently in other classes to develop the concept of volume and to make meaningful the formula for finding the volume.

At the same time this student decided he was going to take all the mathematics he could get (he had not liked mathematics before). His decision is an illustration of what happens to those minds that learn by concrete doing rather than by abstract thinking.

What extra expense will be involved in developing a laboratory? The expense of a laboratory will depend on a number of things. The students and teacher can provide much of the necessary equipment. The expense need not be great. On the other hand, there is much that can be done if money is available.

What equipment is needed? Ample blackboard space is imperative. For graph work one section of the blackboard should be ruled and outlined in pale green or yellow paint, with heavier lines defining every fifth or tenth space. If blackboard space is limited, a substitute is heavy white cardboard, ruled with ink and then covered with a film of Protectophane. If cardboard is the background, wax crayon is used instead of chalk. It can be rubbed off with tissue or a piece of cloth. The squared cardboard can be hung in a convenient spot or rested on the chalk tray. When not in use, it can occupy any convenient place but it should always be accessible.

If the teacher prefers fixed desks and chairs, they should be placed to one side of the room and space left free for group work, conferences, and pupil discussion.

Textbooks other than the regular class textbook should also be available in the laboratory library, since the point of view and methods of different authors should be given consideration. In addition, various reference books, magazines, and other books of cultural and historical value should be found in the library. This library should be a regular part of the laboratory equipment. It may be supplemented by temporary loans from the school or public library.

As materials accumulate, it is necessary to add bulletin boards, bookcases, exhibit shelves, and filing cabinets so that the materials may be used efficiently and saved for future reference. This equipment need not be expensive. In many schools it can be made in the school shops or by interested individual students or teachers. Often there are in storerooms or vacant classrooms unused cases that have long been forgotten and can be used to good advantage here.

Space is needed in the laboratory for the exhibition of posters, clippings, classroom work, etc. A section of wall covered with beaver board is excellent for the purpose. A strip of such material above the blackboard is also a help, as is a swing-wing display board.

The following are illustrations of various types of exhibits:

Mathematics and design. A very interesting exhibit appeared early in the year as the tenth grade pupils were beginning work in constructions. These were changed from time to time, and in the last appearance consisted of a group of the best designs carefully mounted on colored construction paper.

Mathematics and bridges. This material suggests a number of concepts, formulas, and laws of mathematics. It leads to discussions of rigid figures, the catenary, etc.

Mathematics and churches. The solid geometry classes were interested in the shape of the steeples of various churches. At first this included the churches of New Rochelle and later other churches of special interest in our own country and abroad. Then other factors in the architecture and the mathematics involved were noted and utilized in the regular classwork.

Mathematics and defense. This appeared in two separate exhibits. While the upper classmen were working after school hours on an exhibit for *Mathematics and Defense,* an eighth grade pupil became interested not only in the materials but in the mounting of clippings and the way in which they could be mounted. Some time before the upper classmen had completed their work, this boy appeared with a folder filled with carefully prepared clippings and descriptive material. The clippings were concerned with aluminum, petroleum, tin, nitrate, gold, silver, precious stones, silver, etc. These were some of his treasures, saved from his work in the seventh grade. Since all of this had to do with defense, someone asked if he would like to have an exhibit of his own. He was delighted with the suggestion. The exhibit was soon in place and he saw to it that it was a pleasing one. It remained for two weeks as our first defense exhibit. Its owner was surprised to discover that mathematics had to do with these tangible things, but pleased when he found illustrations of mathematics which he could recognize in the mining, transportation, and utilization of these valuable materials.

The materials for the second exhibit on *Mathematics and Defense* were largely made up of pictures from magazines, brought in by many students. Committees worked in mounting and organizing the materials. In one case it was divided into three groups: Land, Sea, and Air. Later, the exhibit was remodeled and soon it became the new exhibit on war.

Mathematics and maps. Even before the exhibit on war had been completed, someone suggested *Mathematics and Maps.* A sheet of paper was posted in a convenient place, and students listed the mathematics they discovered in making and using maps. Discoveries are necessarily always limited by the students' knowledge, but their discoveries help others, give them a certain satisfaction in telling how they made their discoveries, and create interest not only in the specific subject matter but in its utilitarian and cultural values.

One pupil did some research in the making and reading of maps. This provided an opportunity to teach map-making. While there is not always time to follow up all the leads at the time the problem is presented, a challenge is offered that may function in later group or individual discoveries.

Mathematics in the army and navy. This subject, as well as *Mathematics and Aviation,* has been of special interest. Posters showing the importance of mathematics to the armed services were accompanied by pictures of soldiers, sailors, and aviators making maps, reading maps, and surveying. These were used in discussions on "how to study," need for critical thinking, mathematics and reasoning, value of "If . . . Then" type of procedure in our study of mathematics, etc.

Great men of mathematics. This topic adds interest to the history of mathematics. In addition to a general exhibit, the picture of a great mathematician connected with the work to be studied at the time gives realism to the topic. If the picture can be kept in view during the study of the related work and applications, the work relates itself to the great men of mathematics and the time in which they lived. If, before teaching the Pythagorean theorem, a picture of Pythagoras and a short biography are placed where they will be seen, and reference is made to them at the time the Pythagorean theorem is studied, there is aroused a feeling of acquaintance with Pythagoras and with the subject matter.

Several students were interested in history and in the travels of Pythagoras. One drew a map showing where Pythagoras had visited. Another handed in a poem. One wrote a play based on the legend of "The Seven Lamps of Capella Pittagora," in David Eugene Smith's *Poetry of Mathematics;* this was a result of a bit of research done by the class. One group was particularly interested in the Pythagorean theorem and the mathematics necessary to prove this theorem. An illustrated story of the project was the contribution of another student; it included a list of all theorems, constructions, postulates, axioms, and definitions required to prove the theorem. A class in tenth grade mathematics worked on this project while studying the Law of Pythagoras. Other members of the class made interesting contributions to the study. These form but one illustration of the interesting possibilities of the contributions that students can make and the pleasure they can derive from this type of laboratory work.

Another illustration is Archimedes, whose biography and picture are pertinent preceding and during the teaching of ratio and proportion. This furnishes another excellent opportunity for relating science and mathematics. The students of this group were very much interested in discovering how Archimedes determined that Hiero's crown was one hundred per cent pure gold.

The same method may be used in teaching such topics as similarity. Posters illustrating similarity, mounted on the swing-wing display board or any convenient bulletin board, do much to create an interest in similarity if they are in a place convenient for reference at the psychological moment when the subject is introduced. One proof of interest is the new material and posters voluntarily brought in by students.

A further illustration is the use of posters preceding the work on locus. In one case the class had access to a picture of a group of pirates planning a map of the hidden treasure. This picture was taken by one of the students, who had previously built the scene on a table. He had used a piece of tin to represent water; the hills and land were made of sand; trees were bits of evergreen stuck in the sand. The pirates were tiny lead figures. The picture illustrated the story of an original locus problem.

One group was interested in the pantograph, and two boys decided to make one. They did part of their work at home and completed the project in one of the school shops during the directed study part of the mathematics class. The boys had wondered how their instrument would compare with a commercial instrument. One of the girls brought one from home. When the class had completed work with the two instruments, both were given to the mathematics laboratory. It was quite a thrill for this girl to bring an instrument and explain to the class how it was used by an engineer. Work of this kind does much to create a mathematical atmosphere and a will to explore, analyze, and arrive at conclusions.

In a mathematics workshop, posters such as "ANALYZE"—better still, a poster copied from a recent geometry text—are invaluable aids in making analysis a tool of discovery. By frequent reference to these posters the students realize and appreciate the power of analysis and analytic proof.

Three tiny airplane models, each mounted on one of its three axes to illustrate vertical, longitudinal, and horizontal axes, carry the meaning of x, y, and z axes "deep in the heart" of thinking in terms of three dimensions.

In addition to the models, materials, and pictures that are changed frequently, there can be hung on the walls of the laboratory pictures that do much to add to its atmosphere. The pictures on the wall of one laboratory are: "Galileo," a muragraph in soft tones in seven colors; "Tree of Knowledge," "History of Mathematics," "History of the Standard Units of Measurement"—a set of six pictures each 8″ x 10″, which may be framed separately or as one.

Multi-sensory aids contribute much to the process of learning. While some of the values can be measured, there are immeasurable attitudes and responses which are recognized and appreciated by both the teacher and the pupils. David Eugene Smith often spoke of the "soul of mathematics." Possibly the response and attitude that result from this type of study of mathematics are in some way related to the "soul."

A mathematics laboratory is well worth all the effort it takes to build and organize. Multi-sensory aids are part of the commonsense method of developing power to discover, to understand, and to use mathematics and mathematical laws in meeting the challenge of today.

If given a start, a mathematics laboratory is inclined to be a bit like Topsy—to "just grow." However, it does need an opportunity to start. It needs help from administration, and it needs teachers interested in mathematical growth both in learning and in teaching mathematics.

SUGGESTIONS FOR MATERIALS AND EQUIPMENT FOR A MATHEMATICS LABORATORY

FILING CABINETS (vertical), sufficient for filing pupil reports, newspaper and magazine clippings, and picture collections.

POSTER CABINET (or closet), in which large posters and charts may be kept flat.

EXHIBIT CASES for displaying three-dimensional models. (May be old bookcases until better cases are available.)

SWING-WING DISPLAYER, with wings of board. A compact method of increasing available display surface. In theory, this is a large wooden (or fiber board) book, on the "pages" of which pictures and clippings may be displayed.

BLACKBOARD, one section of which is ruled and painted for graph work. (Light green paint is preferable to pure white, and each fifth line should be a bit wider than the others.)

WALLBOARD (for exhibits and display materials), replacing blackboard on one side of the room, and also above the blackboard.

WORK TABLES (and chairs) for discussion and group projects as well as for individual construction work or drawings.

SPHERICAL BLACKBOARD.

GLOBE OF THE WORLD, the type ordinarily used in teaching geography as well as one with markable surface.

SOLID GEOMETRY MODELS, commercial or homemade (regular polyhedrons, conic sections, etc.).

SLIDE RULE (large demonstration model as well as small ones for individual use). (7 ft. Mannheim type, $8.00; Keuffel and Esser Co., 127 Fulton St., New York City.)

OTHER INSTRUMENTS for measuring various units, e.g., barometer, calipers, bushel measure, quart, meter, yard, liter, clock, galvanometer, gyroscope, compass (mariner's), etc.

Transit. Homemade transit is a possibility if a commercial instrument is not available.

Pantograph, not a difficult instrument to make.

Parallelograms and triangles. Made of wooden strips to help teach properties and criteria of a parallelogram; also rigid figures.

Cellophane (cellulose acetate), heavy cellophane (transparent) for making solid geometry models; heavy glazed cellophane for handmade lantern slides. From Celluloid Corporation of America, Newark, N. J.

Protectophane comes in thin, transparent sheets with adhesive on one side. Useful for covering pictures and clippings, and also for covering diagrams or graphs upon which auxiliary lines may be drawn with wax crayons and wiped off again with cleansing tissue. From Stanley Bowmar Co., 2929 Broadway, New York City.

Compasses, protractors, rulers.

Construction paper.

Abacus.

Adding machine and other business machines.

Maps. Write to Superintendent of Documents, Washington, D. C., for Price List 53 *Maps* (Sept. 1940). Lists maps of all sorts at nominal prices.

Wall charts, both commercial and homemade. For suggestions, see *Enriched Teaching of Science in the High School,* by Woodring, Oakes, and Brown, Bureau of Publications, Teachers College, Columbia University; and also *Enriched Teaching of Mathematics* by Woodring and Sanford (same series).

"Tree of knowledge," in color, to hang in the classroom. From Museum of Science and Industry, Jackson Park, Chicago, Ill. 30 cents postpaid.

"History of the standard units of measurement." Set of six pictures, each 8" x 10"; may be framed separately or as one picture. From H. G. Ayre, Western State Teachers College, Macomb, Ill. A set with glossy finish, $2.00; or hand-tinted, $5.00.

"Galileo," muragraph, soft tones in seven colors. From Stanley Bowmar Co., 2929 Broadway, New York City.

Photographs of mathematicians, Portfolio I and Portfolio II. From Scripta Mathematica, Amsterdam Ave. and 186th St., New York City.

Photographs of philosophers (who were also mathematicians). From Scripta Mathematica, Amsterdam Ave. and 186th St., New York City.

Photographs of scientists, 8" x 10". From Science Service, 2101 Constitution Ave., Washington, D. C. Sepia, $1.00 each.

Projectors, opaque, for standard-size lantern slides 3¼" x 4", for 2" x 2" lantern slides and film strips. 16 mm. silent and 16 mm. sound motion-picture projectors. Portable screens and room equipped with dark shades.

SOURCES AND SUPPLIES

Reproducing original charts, posters, and graphs. In any city of moderate size there are companies which make photostatic reproductions at relatively small cost (approx., 25 cents for a sheet 8½" x 11"). This is an excellent way in which copies of interesting and outstanding work by pupils can be kept. Some of these companies also reproduce material in quantity by the photo-offset process (approx., $1.75 for the first 100 copies; 25 cents for each additional 100). Two companies in the New York area are Hudson Blue and Photo Print Co., Inc., 25 Broadway, New York City; Commerce Photo-Print Corporation, 1 Wall St., New York City.

UNIVERSITY PRINTS NOTEBOOK FOR MOUNTING PICTURES. University Prints Co., Newton, Mass. $1.25. This notebook provides space for mounting 200 pictures. The sheets are green-gray. Paper for notes is also provided.

THE PICTURE COLLECTION. John C. Dana and Marcelle Frebault. H. W. Wilson Co., New York City. 90 cents. A paper-bound volume especially designed for librarians, but of value to all teachers. Gives list of publishers and addresses for pictures and postcards. Some of the topics treated are: how pictures are obtained; filing of pictures; mounting, hanging, and display; classification and subject headings.

VERTICAL FILE IN EVERY LIBRARY. Orvitz and Miller. Remington-Rand, Inc., 315 Fourth Ave., New York City. Free. Directions for filing and cataloguing clippings, pictures, etc., which, although primarily for the librarian, are also of value to the mathematics teacher.

THE BULLETIN BOARD AS A TEACHING DEVICE. B. J. R. Stolper. Bureau of Publications, Teachers College, New York City. 20 cents. An interesting account of ways in which this rather neglected teaching device has been used effectively.

GRAPHIC PRESENTATIONS. W. C. Brinton. Published by author, 599 Eleventh Ave., New York, 1939. 512 pp. $5.00. Techniques of making charts, graphs, and other types of representation of facts.

HOW TO USE PICTORIAL STATISTICS. Rudolph Modley. Harper and Brothers, New York, 1937. 170 pp. $3.00. Illustrated with tables, charts, and maps. Useful in all subjects.

GOOD REFERENCES ON PICTURES, MAPS, CHARTS, ETC., as Classroom Aids. E. C. Dent. U. S. Office of Education, Washington, D. C., 1938. 12 pp.

AMATEUR SLIDEMAKING. R. B. Beals. American Photography. Vol. 32, pp. 685-688, October 1938. Valuable suggestions.

HOW TO MAKE HANDMADE LANTERN SLIDES. G. E. Hamilton. Educational Screen, 64 East Lake St., Chicago. 24 pp. 10 cents.

SIMPLE DIRECTIONS FOR MAKING VISUAL AIDS. L. Hethershaw. Department of Visual Instruction, N. E. A., 1812 Illinois St., Lawrence, Kan. 25 cents.

HOW TO MAKE LANTERN SLIDES. W. T. R. Price. Education, Vol. 58, pp. 501-507, April 1938. The method of making lantern slides on glass or lumarith by hand, using a variety of techniques. Directions are also given for a homemade light table for work on slides.

LANTERN SLIDES AND STRIP FILMS. Almost every city of moderate size has at least one company which will make lantern slides, either plain or colored, or strip films, from pictures or negatives. Consult local directory. Among firms in the New York City area are Cosmopolitan Studios, 145 West 45th St.; Edward Van Altena, 71 West 45th St.; Ideal Studios, Inc., 160 West 46th St.

CARE OF FILMSLIDES AND MOTION-PICTURE FILM IN LIBRARIES, Superintendent of Documents, Washington, D. C., 1936. 8 pp. 5 cents. Standards Research Paper No. 942, on the care and storage of this visual material.

BESELER LANTERN SLIDE CO., 131 East 23rd St., New York City. This company sells lantern slide materials.

MEASUREMENT

HISTORICAL REVIEW OF THE MEASUREMENT OF LENGTH, Ford Motor Co., Dearborn, Mich. Free. A folder giving a short illustrated history of measurements.

STANDARD MEASUREMENTS OF THE INTERNATIONAL METRIC SYSTEM. U. S. Bureau Department of Commerce, Miscellaneous Publications No. 64. Superintendent of Documents, Washington, D. C. 34 pp. 15 cents. Although published in 1935, this bulletin is still one of the most valuable publications in the field.

STANDARD MEASUREMENTS OF THE INTERNATIONAL METRIC SYSTEM. U. S. Bureau of Standards. Superintendent of Documents, Washington, D. C. 40 cents. A valuable wall chart.

STANDARDS OF WEIGHT AND MEASURE. Price list No. 64. Superintendent of Documents, Washington, D. C. Free. A list of government publications dealing with tests of metals, cement, concrete, iron, electricity, clays, and photographic materials. Many publications on standards of weight, length, electricity, resistance, etc.

EUGENE DIETZGEN CO., 218 East 23rd St., New York City. Pantographs, graph paper, and similar supplies. Write for further information and samples.

DEMONSTRATION SLIDE RULES. Keuffel and Esser Co., 127 Fulton St., New York City. Mannheim type (No. 100) 7 ft. long, $8.00. Polyphase type (duplex) 7 ft. long, $15.00.

YODER INSTRUMENTS, East Palestine, Ohio. Mathematical aids.

VISUAL AIDS IN TEACHING

E. R. Breslich

When visual aids are used in teaching, it is as necessary as in the case of any other teaching device to have clearly in mind the objectives to be attained with the pupils. Faulty use of a teaching procedure makes teaching ineffective. It may even destroy what the procedure is expected to accomplish. For example, one of the aims of supervised study is to make the pupil intellectually independent; yet one of the most common criticisms of supervised study is that it makes pupils depend too much on the teacher. The teacher who thinks that he can help the pupil best and most quickly by removing all obstacles will be doing much of the work for him and thus training him to be more and more dependent. However, the teacher who stimulates interest and effort, raises questions for which the pupil finds the answers, and instructs him in effective ways of study will train him to become more and more independent. Thus, the criticisms are not really directed against supervised study but against the way of administering it.

Similarly, those who minimize the importance of visual aid or even condemn its use have probably been influenced by faulty uses in teaching which they have observed.

The following discussion of the more important objectives and of the methods of using visual aids may suggest ways of making them effective.

Understanding the Meanings of Mathematical Concepts

Careful analyses of the mental processes and of the written work of pupils disclose that much of the difficulty they experience in the study of mathematics is caused by the failure to understand the basic mathematical concepts. Unfortunately, these concepts are too often regarded as simple by the teacher. For example, children are taught fractions in the upper grades of the elementary school and use them throughout high school. However, the errors which high school students make are convincing evidence of their lack of a clear understanding of the meaning of the fraction concept. The same is true of other basic concepts in the various mathematical subjects; for example, the literal number, signed number, exponent and equation in algebra; the triangle, circle, and area in geometry; and the trigonometric ratio and function in trigonometry. The teacher in search of ways of helping the pupil attain the necessary understandings will find that visual aids are a most helpful ally.

The following illustrations indicate the procedures for using visual aids with several basic mathematical concepts.

Literal number. The introduction of letters as number symbols marks an important step in mathematical education. Three aspects need to be taught in the beginner's course.

First, the idea is to be developed that a literal number is an unknown number which is unlike the specific numbers of arithmetic. The discussion may start from a line segment, or an angle, of unknown length or size. Conveniently this may be developed by a letter, such as l or a. The pupil then determines by measurement the numbers denoted by the letter l or a. Thus he learns how letters are used conveniently as symbols for unknown numbers whose values may be found by some process, in this case by measurement. The mistake of thinking of literal numbers as names is thereby avoided. From the beginning they are actually symbols for numbers.

A second aspect leading the pupil to a fuller understanding of the meaning of the literal number concept is the idea of general number, that is, of a number symbol which may have any value whatsoever. This may be derived from a triangle whose unknown perimeter is denoted by $a + b + c$. Measurement then determines the value of the unknown number $a + b + c$. However, the triangle may be any triangle, and $a + b + c$ therefore denotes the perimeter of all triangles that may be drawn. It will have a different value for each new triangle; in fact, it may have an indefinite number of values.

The idea that a literal number may be a variable number is visualized with the number scale. When a train has traveled a given number of minutes, the distance passed over may be denoted by *d*. As the train continues to travel and the number of minutes changes, the distance represented by the literal number *d* also changes. The changes may be visualized by marking various values of *d* on the number scale. The number *d* is said to "vary" with the number of minutes. It is a variable number.

Angle. Evidence that many pupils who have studied geometry for some time are confused about the meaning of an angle is obtained when they compare sizes of angles on the basis of lengths of sides or when they speak of a straight angle as a straight line.

The first step in developing this concept is to identify illustrations of angles in the classroom and out of doors, such as those observed on walls, ceiling, floor, desk, chalkbox, and other familiar objects found in every classroom. From such observations the pupil comes to the conclusion that an angle is a geometric figure formed by two straight lines which meet or intersect.

The blackboard compasses may then be used to visualize the meaning of the size of an angle. Keeping one arm of the compasses fixed in position, the teacher turns the other arm so that it occupies several other positions. This illustrates that the size of an angle depends on the size of the opening of the compasses or on the amount of turning, not on the lengths of the sides. The smaller of two angles may actually have the longer sides. The blackboard protractor is then introduced to show how the size of an angle may be found by measurement. The same instruments are used with good results to clarify the meanings of acute angle, right angle, obtuse angle, and straight angle. A straight angle is something very different from a straight line, because it has a vertex and two sides. It is found by turning one side to the position of the other, not by moving the pencil point along the edge of a ruler.

Signed numbers. Experience has shown that pupils have much difficulty with signed numbers. Because of their importance in all algebraic work the best teaching procedure should be employed. Teachers have used with success a variety of devices to develop clear meanings. One of the simplest and most effective is the number scale, with which the pupil is usually familiar from its uses in measurement. He is acquainted with the fact that, as he passes from the left to the right, the numbers on the scale increase, and that as he passes from the right to the left, the numbers on the scale decrease to zero. The scale is now extended to the left beyond the zero mark, and points are marked one, two, three, and more units from zero. They are labeled -1, -2, -3, etc. The entire scale is now examined to bring out such facts as the following:

(a) All negative numbers are less than zero, in the sense that they lie to the left of zero. Thus, $-1<0$, $-2<0$, $-3<0$, etc.
(b) Any number on the scale is less than any number to its right; e.g., $-3<-2$, $-2<-1$, etc.
(c) Any number is greater than any number located to its left; e.g., $-3>-5$, $0>-8$, etc.

The number scale need not be the only device used in teaching signed numbers. Temperature readings, distances north and south or east and west, and other devices should follow until the meaning of positive and negative numbers is clearly understood. Not until this has been accomplished is the pupil ready to take the next step, to learn how to operate with signed numbers.

Equation. One of the best devices used to lead the pupil to a full knowledge of equations is a pair of scales such as are used in laboratories. When in balance, the total amount is the same on both sides. In that position the scales illustrate an equation. Thus an equation is a statement of the equality of two number expressions.

If the total amount on one side of the scales is changed, an equal change must be made on the other side to preserve the balance. Thus we may add the same number of weights to both sides, take away (subtract) the same number from both sides, double or treble (multiply) the number of weights on both sides, or replace them by one-half, one-third, etc. (divide) of the number of weights.

The changes with the loadings of the scales visualize the basic principles employed in solving equations. The process changes the values of the members of the equation, but does not destroy the equality.

By always thinking of an equation as a statement of balance of two number expressions, the pupil will escape much of the confusion into which he is thrown by such mechanical processes as transposition. Solving an equation always means to add to, subtract from, multiply, or divide both members by the same number.

Trigonometric function. For a complete understanding of the trigonometric functions the pupil must master such fundamental facts and principles as the following:

(a) A knowledge of the changes in value of the function as the angle changes from 0° to 360°.
(b) A knowledge of the greatest and least values which a function may have.
(c) Ability to determine the algebraic sign of a function for all angles lying in a particular quadrant.
(d) Ability to express the function of any angle in terms of an angle less than 45°.
(e) Ability to express the values of functions of negative angles in terms of positive angles.

These facts and others may be developed separately, but they can all be visualized by one device, the graph. A quick glance at the graph, or indeed a mental picture of it, enables the pupil to answer questions relating to any of the facts stated above. The time used in making and interpreting the graphs of the trigonometric functions is very profitably spent.

UNDERSTANDING MATHEMATICAL PROCESSES

Such abstract processes as division of decimal fractions, extraction of square root, factoring, and the operations with signed numbers can be taught by rules. However, a rule is gradually forgotten and, in the performance of a process, confusion arises concerning the steps involved and the order in which the steps are to be taken. The difficulty may be reduced or even eliminated by the use of visual aids to illustrate the steps in a process and to clarify the reasons for the various steps. The pupil thus taught is given a chance to return to these aids and through them to rise above the confusion in which he finds himself.

The following examples show the use of visual aids in teaching mathematical processes:

Addition of signed numbers. Two rules are usually taught: one applies to numbers having like signs; the other, to numbers having unlike signs. The number scale visualizes both cases. The rule in the second case consists of the following steps. The arithmetical difference is taken, the sign of the number having the larger absolute value is fixed in mind, and that sign is written before the arithmetical difference. If a pupil has forgotten the rule or if he is not sure of it, he may return to the number scale to find the answer, or he may use the number scale to reconstruct or verify the rule. To find the sum of $+12$ and -3 he proceeds as follows: $+12$ added to -3 means that he is to locate $+12$ on the scale and from that point to move 3 units to the left. He is then 9 units to the right of the zero mark, which means that the answer is $+9$. It is not necessary to make a drawing of the scale, because the steps are so simple that they are easily visualized mentally.

By this device he may add two numbers having like signs. Subtraction may also be visualized on the number scale. In this case the scale is merely used to visualize and develop the rule. Because the rule is simple, pupils do not need to use the scale to solve subtraction problems.

Finding the square root of a number. Because this process is not used as frequently as the four fundamental processes, it is more easily forgotten. Hence, it is necessary to use a method of teaching which will facilitate the recall of the steps involved.

The given number whose square root is to be found is represented by the area of a square composed of two squares whose areas are a^2 and b^2, and two equal rectangles each of area ab. The area of the large square is $a^2 + 2ab + b^2$, which therefore represents the given number. This suggests the steps to be taken in finding the side $a + b$, which represents the required square root. The problem thus reduces itself to finding a and b. The steps are:

(a) To find the largest number a whose square is less than the given number, i.e., $a^2 + 2ab + b^2$.
(b) a^2 is subtracted from the given number, i.e., $a^2 + 2ab + b^2$.
(c) The remainder $2ab + b^2$ shows the remaining steps, i.e., to divide it by $2a$ to determine b, and to subtract $2ab + b^2$ from the previous remainder.
(d) If there is no new remainder, $a + b$ is the required root.

If there is a new remainder after subtracting $2ab + b^2$ or if the given number is not a square, steps similar to (c) are repeated.

The purpose of the use of the square is to facilitate the recall of $a^2 + 2ab + b^2$, which then shows the steps to be taken in the process.

Solving equations. It has been shown above how a pair of scales may be used to visualize the meaning and the process of solution of simple equations. As the pupil continues the study of algebra, other difficulties with equations arise. For example, it is not easy to see why a quadratic equation has two roots that are equally valid, why the solution of a system of linear equations should consist of a pair of numbers, and why simultaneous quadratic equations should have more than one pair of numbers satisfying the equations.

The answers to these perplexing problems are found in the use of graphs to visualize the processes of solving and in the solutions themselves. The graph of a quadratic function $ax^2 + bx + c$ shows the changes in the values of the functions. When the graph crosses the x-axis twice, the function assumes the value 0 twice; the equation $ax^2 + bx + c = 0$ has therefore two roots. The graphical method also shows that four solutions are to be expected in the case of simultaneous quadratics and that they are not necessarily distinct or even real. Furthermore, it illustrates each step in the process of solving and the meaning and purpose of each new equation derived from those that precede. It explains why the final results are the solutions of the original equations.

Experience has shown that the confusion that arises in the study of equations may be avoided by letting the graphical solutions always precede the algebraic methods.

UNDERSTANDING RELATIONSHIPS

Mathematical relationships are more readily understood and permanently retained if they are illustrated in diagrams and models. The pupil who associates with the expression $(a + b)^2$ the area of a square which is divided into two squares and two equal rectangles the sum of whose areas is $a^2 + 2ab + b^2$ will always remember the correct equivalent of $(a + b)^2$. If the area of a triangle is developed from a diagram showing it to be one-half of the parallelogram, the pupil will understand why the factor $\frac{1}{2}$ occurs in the formula $A = \frac{1}{2}bh$. Similarly, he will not forget that the formula for the volume of a pyramid contains the factor $\frac{1}{3}$ if he has developed it by pouring the contents of a pyramid filled with sand into a prism having the same base and altitude as the pyramid and finding that he must do it three times in order to fill the prism. He will be able to recall the formula $V = abc$ for finding the volume of the rectangular parallelepiped if he has handled a rectangular block divided into a layers, each of which is divided into b strips, each of which in turn is divided into c unit cubes. He will produce automatically the relationship between the sides of a right triangle if he visualizes the familiar figure of a right triangle with a square drawn on each side.

Teachers recognize the importance of the ability to visualize relationships in algebra and analytic geometry. Pupils are expected to know that the graph of $y = mx + b$ is a straight line, that the graph of $y = ax^2 + bx + c$ is a parabola, and that the points of intersection illustrate the solutions of the system

$$\begin{aligned} ax^2 + bx + c &= 0 \\ mx + b &= 0 \end{aligned}$$

Thus they recognize the fact that a pair of equations, one of which is linear and the other quadratic, cannot have more than two solutions. These and further important facts which are not readily understood by solving the equations may be derived from graphs. Similarly, pupils are expected to know that the ellipse, the hyperbola, and the circle represent, respectively, the relationships $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, and $x^2 + y^2 = r^2$. Such knowledge is useful in the solution of simultaneous equations.

TRAINING IN SPACE IMAGINATION

Training in space imagination is generally listed as a major objective of the teaching of geometry. Training is offered in plane geometry by extending the ideas and relations in the plane to three-dimensional space where they may be observed in familiar settings. The course in solid geometry aims further to develop the ability to visualize spatial figures and relations. It also trains pupils to make good drawings of these figures on paper. Furthermore, pupils should learn to visualize in three-dimensional space the points, lines, and planes shown in two-dimensional drawings. Very often herein lies a major difficulty in teaching solid geometry. Some pupils require considerable training before they are able to visualize from the textbook drawings the geometric forms which they find in their surroundings. Models of the drawings are helpful, especially those made by the pupils themselves.

It is advisable that in the early stages of the course, the study of the diagrams in the textbook be preceded by an examination of models. Thus, the pupil may count the vertices, edges, and faces of a cube on the tangible model. He may note the relative positions of parallel and perpendicular lines and planes, and he may derive therefrom the formulas for finding the areas of faces and of the entire surface.

When he has become familiar with the model, he is ready to learn to draw it on paper. Many pupils need time and practice before they can make a satisfactory drawing. After that, they are prepared to study profitably the diagram of the cube in the textbook. Even then some pupils will find it necessary, or helpful, to return to the model to clarify their ideas.

Frequently, before reciting in class, the pupil may build up a model of the diagram with sticks and cardboard. Wide opportunity is offered here for the use of creative imagination.

Some teachers object to the use of visual aids because they believe that the time spent on making models and building up diagrams could be more profitably employed in the study of the geometry involved. It is true that some teachers and pupils do become so interested in this type of work that they carry it to extremes. In such cases the study of geometry might actually suffer. Most teachers, however, will avoid extremes, especially sin[illegible] right kind of training will reduce the need for models [illegible] the point where they will be used only in cases which experi[illegible] shown to be very difficult. This may be necessary even late in the course. For example, few pupils are able to form a clear conception of spherical triangles and polygons merely by looking at the diagrams usually found in textbooks. If they are permitted to observe how the teacher draws a spherical polygon on a spherical blackboard, the difficulty is easily removed. If no spherical blackboard is available, good results may be obtained with a baseball, an orange, or an apple. Photographic views of polygons drawn on a spherical blackboard have been used with good results.

INCREASING THE PUPIL'S UNDERSTANDING OF HIS ENVIRONMENT

One of the objectives of education is to acquaint the pupil more fully with his surroundings. Mathematics may contribute to this knowledge by making use of the many opportunities which geometry offers for observing geometric facts and principles in the classroom and out of doors. It is not difficult to find evidence that such training is badly needed. Some pupils cannot supply even the most elementary information about the height of the building in which they live, the size of the lot on which it is built, the width of the street, and the distance to school. Partly responsible for this lack of observation are the teachers of mathematics who limit their teaching to the facts in the textbook and fail to call attention to the abundance of illustrations in the familiar forms and objects which pupils may observe all around them.

The right kind of teaching makes pupils geometry-minded, so that they recognize that geometric forms are a part of their surroundings. The windowpanes and blackboard represent plane surfaces. Triangles are discovered whose sides are flagpoles or smokestacks. Filing cases, railroad cars, trucks, and cartons are rectangular forms. Fruit cans, boilers, tree trunks, and silos are cylinders. Tumblers, wastebaskets, posts, and lampshades are frustums of cones. Everywhere the pupils may recognize illustrations of the figures which they study in geometry.

A small amount of simple work in surveying should replace some textbook problems to increase the pupil's appreciation of mathematics.

CLASSROOM EQUIPMENT

In most schools the mathematics classroom has poor equipment when compared with that of other departments, e.g., the sciences. For years it has been the opinion of school administrators in general that mathematics teachers can get on very nicely with an eraser and a straightedge. It is difficult to change this attitude, which fails to recognize that mathematics courses can be greatly enriched by the use of equipment which relates mathematics to everyday life and which, by illustrating and visualizing mathematical facts and principles, increases the pupils' understandings.

The following list of visual aids is by no means complete but it should prove helpful.

Materials present in the classroom. The fact that certain material is available without cost does not minimize its value. Indeed, its simplicity and familiarity make it very effective, particularly in developing the basic concepts of geometry. The teacher should refer freely to illustrations found on the walls, ceiling, and floor of the room, the teacher's desk, chalkbox, windows, doors, bookcases, and filing cases.

Materials donated or made by the pupils. Here one may include sheets of cardboard, sticks, string, pictures, posters, homemade transit, plane table, and models. With a cardboard tube and two small protractors a skillful pupil can make a transit which is quite satisfactory for simple work in surveying.

In some schools teachers and pupils have in a short time assembled excellent collections of posters which illustrate mathematical principles or show the uses of mathematics. Recently such a collection of pupil-made posters was exhibited by Miss Ida Fogelson, a teacher in the Chicago school system.

Patterns and instructions for making cardboard models of polyhedrons can be purchased from M. C. Hartley, a teacher in the University High School, Urbana, Illinois. It is surprising how this type of work stimulates the creative imagination of pupils. Each new class will try to surpass the preceding class. If only the best are retained, an excellent permanent collection of models may be accumulated in a comparatively short time. The next article, by Dr. Hartley, explains how much of this work can be done.

Inexpensive but essential equipment. Here one should list blackboard rulers, compasses, protractors, colored crayon, a spherical blackboard to visualize spherical angles and polygons, a squared blackboard for graphical work to clarify mathematical ideas and processes.

Pictures of bridges, dams, and buildings exhibiting mathematical facts show the applications of mathematics in art, nature, and architecture.

More expensive equipment. This may include low-priced transit, steel tape, level, and sextant for doing some field work to supplement geometry and trigonometry, diagrams in three dimensions, such as those published by Newson and Company, New York City, to illustrate and emphasize the significance of mathematics in everyday life, stereoscope with views showing the practical applications of mathematical principles.

Films are a valuable instrument for teaching mathematics. In their use mathematics is far behind the sciences. It is to be hoped that this type of visual aid to mathematics will be more fully developed in the near future.

A LABORATORY APPROACH TO SOLID GEOMETRY

Miles C. Hartley

DURING RECENT YEARS emphasis on logical reasoning in everyday life and on practical applications, as well as innovations to provide for individual differences, has resulted in radical changes in the courses of study and in the textbooks in plane geometry. Solid geometry, however, has not enjoyed this renovation; "Solid Geometry for Today" and "New Solid Geometry" are but modern names for *Euclid,* Books VI to IX, with the result that solid geometry as now taught in our secondary schools has been called the most uninteresting subject of the curriculum. It is the writer's purpose to present here the results of experiments conducted at the University High School, Urbana, Illinois, in the hope that this presentation may lead other teachers to try more laboratory work in their teaching of solid geometry and thus to stimulate greater interest in the subject.

One of the primary objectives of a course in solid geometry should be the development of a pupil's geometrical imagination so that he may visualize clearly the correct spatial relationships. This mental growth may be speeded up through perspective drawings and pupil-made models.

PERSPECTIVE

The ability to look at a two-dimensional drawing of a three-dimensional object and to see it as a solid figure can, by a study of perspective, be developed so that diagrams of planes and polyhedrons assume a reality of depth as one reads into the figure the important third dimension.

The means of such development here presented is a lecture-laboratory method in which the teacher makes the drawings on the blackboard as he gives the explanation, and the students make the same drawings at their desks. The following discussion introduces the students to solid geometry through perspective drawing.

"In plane geometry you studied many figures, such as the polygon and the circle, which can be represented on a flat surface called a plane. These figures had only two dimensions: length and width. In solid geometry you will study figures of three dimensions: length, width, and thickness. These figures, such as the cylinder, cone, prism, and pyramid, are not confined to a plane; hence, they differ from the figures of plane geometry. Before you study the properties of these figures, you will want to learn to draw them so that they will appear to be solid and not flat like a sheet of paper. This is done by a method called *perspective*, which is used to make an object appear to have dimensions."

"If you were riding along a concrete highway in the level country of the Middle West, you would see the flat prairie stretching away in the distance until it seemed to meet the sky in a long, straight line which is called the horizon. In front of you, as you ride along, the wide concrete highway goes on and on, getting apparently narrower and narrower, until it meets the horizon in a single point. This point is called the vanishing point. If you looked very carefully, you would discover that the horizon seemed to be the same height as your eyes. If you left your car and sat on the ground, you would find that the horizon had lowered. If you climbed a near-by tree, you would find that the horizon rose as you climbed. Let us draw these three situations (Fig. 1)."

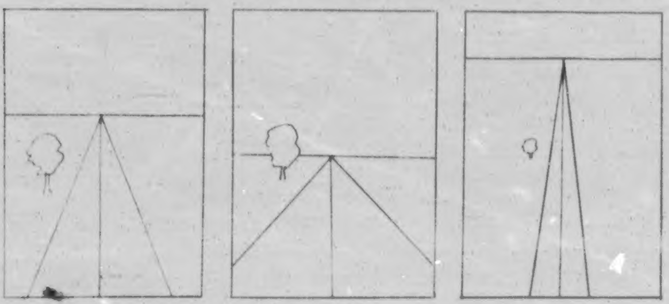


Figure 1

(*Exercises.* Make three drawings of railroad tracks in which the horizon is at three different eye levels. Bring to class examples of vanishing points in pictures and photographs.)

"Can you imagine a large box placed on the highway? The edges of the box which rest on the highway form parts of lines which, if extended along the highway, would meet at the vanishing point. This would likewise be true for the upper parallel edges. In Figure 2 you have a sketch of the box as it would appear if you were standing beside it. Figure 3 shows how the box would appear if you were seated before it."

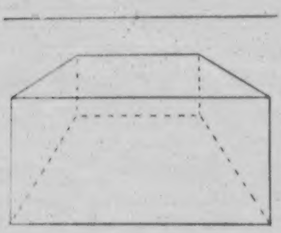


Figure 2

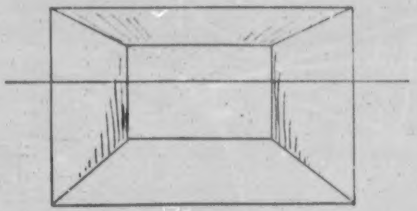


Figure 3

(*Exercises.* Draw a kitchen table as it would appear if you were standing beside it. Draw the table as it would appear if you were sitting beside it. Draw a picture of a stretch of a railroad track with telegraph poles and fences on either side of it.)

"Parallel lines are lines that always remain the same distance apart and hence do not intersect no matter how far they are extended. In Figure 2, four edges of the box are parallel and all seem to meet at the vanishing point. As we look down the highway, the sides of the concrete roadbed, the rows of telephone poles, the telephone wires, and the fence form sets of parallel lines which meet at the same vanishing point. Hence, in perspective, a group of parallel lines which recede from the observer converge at a vanishing point on a level with the observer's eye. The exceptions are parallel lines which run from right to left (parallel to the horizon) and parallel lines which are vertical. In the box in Figure 2, there are four sets of each type of parallel lines."

"The ordinary building brick (called a rectangular parallelepiped in solid geometry) has six rectangular faces; its edges form three sets of parallel lines—four parallel lines of width, four parallel lines of length, and four parallel lines of height. If you turn the brick so that you look along the length lines, you have an end view of it (Fig. 4). The vanishing point does not appear in the drawing, but you can find it by extending the two visible lines which represent the parallel edges of length (Fig. 5). The horizon is drawn through this point and parallel to the front edges of the brick. When the brick is placed so that three of its faces are visible, there are three lines which can be extended to locate the vanishing point and the horizon. As the brick is rotated clockwise, the vanishing point moves farther to the right (Fig. 6, 7, and 8)."

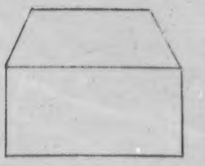


Figure 4

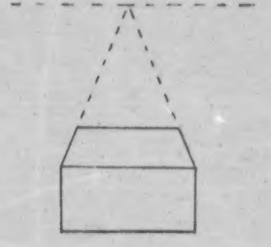


Figure 5

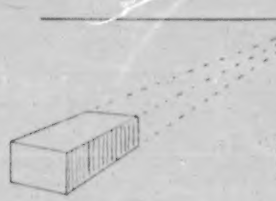


Figure 6

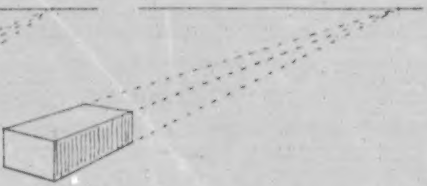


Figure 7

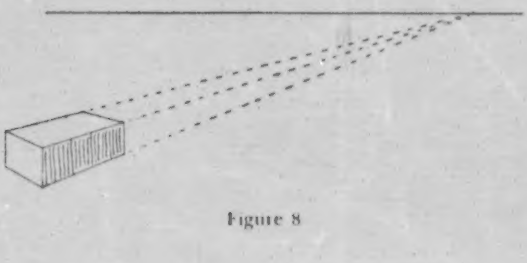


Figure 8

"You must not forget the parallel lines which represent the width of the brick. When you extend these lines, you find a second vanishing point which lies on the same horizon as the first vanishing point (Fig. 9). As you turn the brick more and more, you find that the first vanishing point moves away to the right along the horizon and that the second vanishing point moves to the right also. The third set of parallel lines is drawn straight up and down; there is no vanishing point to consider for them."

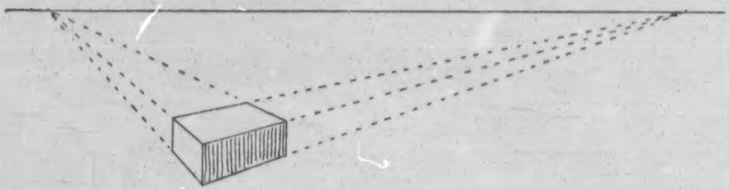


Figure 9

(*Exercises.* In a photograph locate the vanishing points by extending parallel lines. Make a drawing of a cube in each of the following positions: (a) midway between two vanishing points and below the horizon; (b) to the right-center of two vanishing points and below the horizon; (c) to the left-center of two vanishing points and below the horizon; (d) midway between two vanishing points on a horizon behind the cube; (e) to the right-center of two vanishing points on a horizon behind the cube; (f) to the left-center of two vanishing points on a horizon behind the cube.)

"The center of a rectangle is the point of intersection of the two diagonals (Fig. 10). The perspective of a rectangle is a quadrilateral, but its center is still the point of intersection of the diagonals. Thus you have a method for determining the point of bisection of a line segment in a perspective drawing."

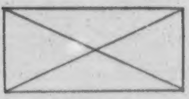


Figure 10

"*ABCD* (Fig. 11) is the perspective of a rectangle. *E* is the intersection of diagonals *BD* and *AC*. *E* joined to the first vanishing point V_1 determines the points *H* and *I*, which are the mid-points of *DC* and *AB*, respectively. Similarly, *E* joined to the second vanishing point V_2 determines *F* as the mid-point of *AD*, and *G* as the mid-point of *BC*."

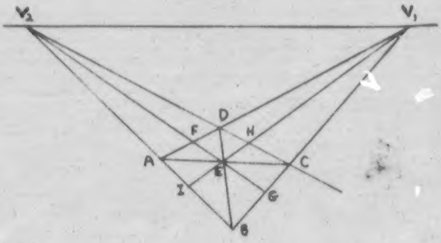


Figure 11

"You can use this same method to determine the fourth side of a rectangle when you know three sides and the center. For example, *AB* and *CD* (Fig. 12) are two parallel lines of the same length; *E* is the mid-point of *AB*. Draw the three parallel lines *AC, EF,* and *BD*. The line *AF* will meet *BD* extended in point G (Fig. 13); *G* determines the position of line *GH*, the fourth side of the rectangle of which *F* is the center (Fig. 14)."

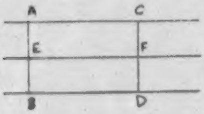


Figure 12

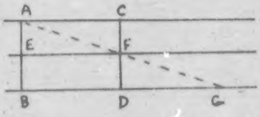


Figure 13

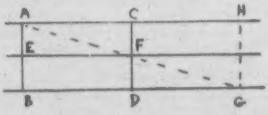


Figure 14

"This same relationship holds true in perspective and enables us to draw to scale. Draw any vertical line *AB* with mid-point *E* (Fig. 15). Draw the parallel lines *AV, EV,* and *BV*. On *BV* lay off

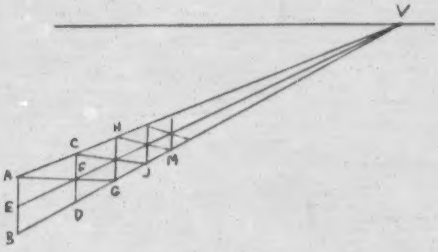


Figure 15

BD, the desired unit of length. Draw *DC* parallel to *AB; CD* intersects *EV* in *F; AF* intersects *BV* in *G;* draw *GH* parallel to *AB*, etc. Then $BD = DG = GJ = JM$, etc. Now, if you wish to draw the perspective of a fence and have two posts placed, you can draw as many posts as you wish all correctly spaced."

(*Exercises.* Draw a picture of railroad tracks with telegraph poles on either side. Draw the perspective of a checkerboard eight units on a side (a) using one vanishing point and (b) using two vanishing points.)

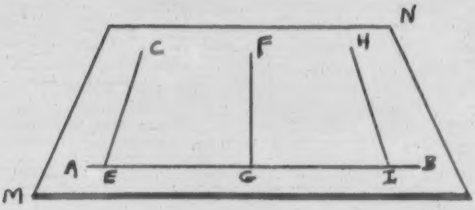


Figure 16

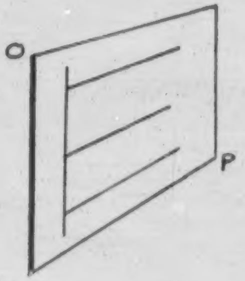


Figure 17

"In plane geometry we spoke of a plane as a flat surface. Now in solid geometry we shall need a more precise definition expressed in mathematical language. Hence, a *plane* is a surface such that, if any two of its points are connected by a straight line, the line lies entirely in the surface. Since there is no limit to the size of a plane, it is impossible to show a complete plane in a drawing. It is customary to think of a plane as a rectangle which in a drawing becomes a trapezoid or a quadrilateral, depending upon the position of the vanishing point. *MN* is a horizontal plane (Fig. 16) and *AB, CE, FG, HI* are lines in *MN. CE, FG,* and *HI* are all perpendicular to *AB*. *OP* is a vertical plane (Fig. 17)."

(*Exercises.* Draw a right triangle in a horizontal plane; an isosceles triangle; an equilateral triangle. Draw two planes intersecting in a horizontal line; in a vertical line.)

The course may now proceed in the customary manner with the introductory work on lines and planes. It would be a grave error, however, to terminate all discussion of perspective at this point. Perspective drawing should be used throughout the course and further explanations of the drawing of prisms, pyramids, cylinders, and cones should be given as needed.

SIMPLE MODELS

Models have been employed for many, many years, but they have not been used extensively in solid geometry because their value in the development of the concept of space has been woefully underestimated. In the early stages of a course in solid geometry, models are essential. Sets of little wooden models of geometrical solids are available commercially but are too small for classroom demonstration; hence, homemade models as here described are usually to be preferred.

The elementary relationships of lines and planes are easily depicted, as is shown in the following examples where definitions, postulates, and theorems concerning lines and planes are illustrated by means of cardboard, wire, cork, and string. The cardboard should be quite heavy, or thin boards or pieces of plywood could be used. The wire could be pieces of an old clothes hanger.

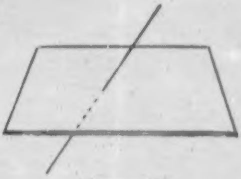


Figure 18

A straight line intersects a plane in a point. Use a piece of cardboard to represent the plane and a piece of wire to represent the line (Fig. 18).

Two planes intersect in a straight line. Cut two pieces of card-

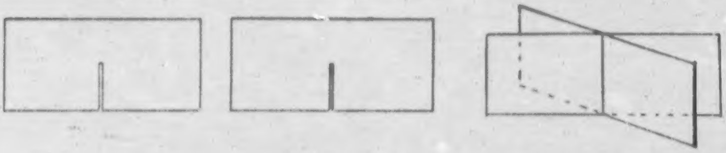


Figure 19

board halfway through and place them together to form the intersecting planes (Fig. 19).

Three planes may intersect in three parallel lines. Cut three pieces of cardboard halfway through and place them together to form the intersecting planes (Fig. 20).

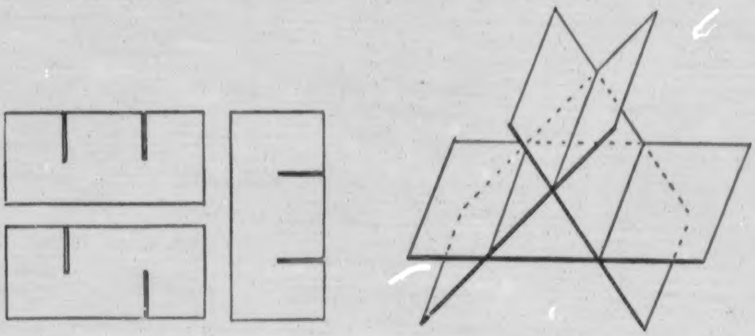


Figure 20

Three planes may intersect in a straight line. Cut three pieces of cardboard halfway through as shown (Fig. 21).

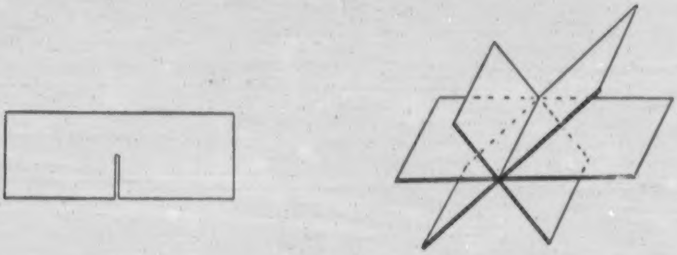


Figure 21

Three planes may intersect in a point. Join three triangles as in Figure 22. $AB = AF$. Cut out around the edge, leaving a lapel on AF. Fold on AF, AE, and AC; paste the lapel on AB.

If two straight lines do not lie in the same plane, they are skew lines. Use four pieces of wire and three corks (or modeling clay). Figure 23 shows two skew lines perpendicular to two intersecting lines; $AE \perp AD$, $CB \perp DB$.

Each of three lines may be perpendicular to the other two. Use three pieces of wire and a cork. *AB, CB,* and *DB* are mutually perpendicular (Fig. 24).

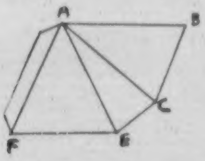


Figure 22

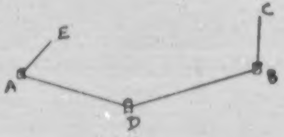


Figure 23

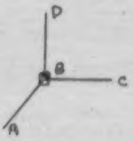


Figure 24

Each of three planes may be perpendicular to the other two. Use three pieces of cardboard cut as shown in Figure 25.

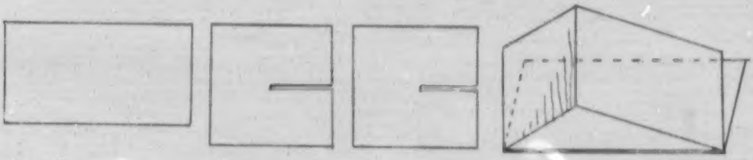


Figure 25

All the perpendiculars to a line at a given point lie in a plane perpendicular to the given line. Use cardboard, cork, and several pieces of wire. The vertical line is the given line; other wires are perpendicular to it. The cork holds the wires and is glued to the cardboard, which represents the plane in which the perpendiculars lie (Fig. 26).

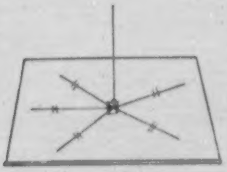


Figure 26

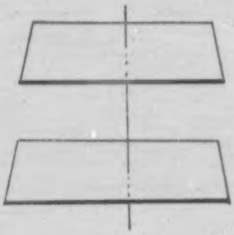


Figure 27

A line perpendicular to one of two parallel planes is perpendicular to the other. Use two pieces of cardboard and a quarter-inch dowel. Glue at the points of intersection (Fig. 27).

If two parallel planes are intersected by a third plane, the lines of intersection are parallel. Use three pieces of cardboard cut halfway through, as shown in Figure 28.

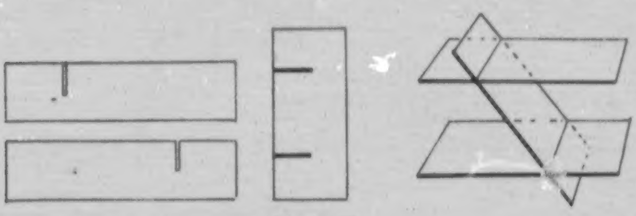


Figure 28

If a line is perpendicular to each of two intersecting lines, it is perpendicular to the plane of the two lines. A quarter-inch dowel (or piece of wire), a piece of heavy cardboard (or plywood), and

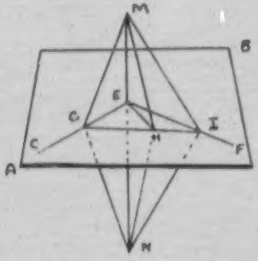


Figure 29

string are the materials needed. AB, the cardboard, represents the plane; MN, the dowel, represents a line perpendicular to the lines CE and EF drawn on the cardboard. $ME = EN$. GM, HM, IM, GN, HN, and IN are string (Fig. 29).

STRING MODELS

It is rather generally assumed that after the introductory unit in solid geometry the average pupil should need no further assistance in the visualization of space relationships. But many of the difficulties which arise can be traced directly to the pupil's inability to "see" the figures. For this reason it is desirable to make some use of models throughout the course.

In units on Prisms, Cylinders, Pyramids, and Cones many theorems can be readily illustrated by means of string models. A brief suggested list of such theorems follows:

(a) The sections of a prism made by parallel planes cutting all the lateral edges are congruent polygons.
(b) The plane passed through two diagonally opposite edges of a parallelepiped divides it into two equal triangular prisms.
(c) Every section of a cylinder made by a plane passing through an element is a parallelogram.
(d) If a pyramid is cut by a plane parallel to the base, the section is similar to the base.
(e) The volume of a triangular pyramid is equal to one-third the product of the altitude and the area of the base.
(f) The intersection of a right circular cone and a plane is a conic section.

Figures, two parts of which lie in parallel planes, are the easiest to represent as models, but more complex ones can be reproduced. One of the most valuable models is the one for the theorem, *An oblique prism is equal to a right prism whose base is a right section of the oblique prism and whose altitude is equal to a lateral edge of the oblique prism*, in which the base of the oblique prism is not parallel to the base of the right prism. Practically every theorem and corollary represents a possibility for conversion into concrete form by means of wood and string.

The materials needed for string models are plywood, one-quarter of an inch thick and cut into foot squares; dowels, three-eighths of an inch in diameter and cut into foot lengths; balls of carpet warp (or thread) in several colors; three-quarter inch nails. The average cost per model is about eleven cents. Figure 30 shows a model for theorem (f) where the conic section is a hyperbola.

The ingenious teacher will find many ways to introduce this work. One possible procedure would be a unit on Models to follow the unit on Lines and Planes. First, it would be necessary to define polyhedron, prism, parallelepiped, pyramid, cylinder, and cone. These definitions would take on deeper meaning if the solids were actually constructed to meet the given qualifications.

For example, in the case of a parallelepiped, three pairs of congruent parallelograms of correct dimensions could be fastened together by means of gummed labels; in the case of a cylinder, the solid could be generated by rotating a rectangle and then constructed from a rectangle and two circles of correct dimensions. Second, as soon as these definitions were understood, a list of theorems similar to the one above would be presented to the class and each member would choose a theorem for which he wished to make a model. Third, Laboratory Sheet No. 1 (see page 55) could be distributed and the directions there could be followed until the model was complete. The actual construction might be made in the woodshop if one were available, but the work is so simple that it could be carried out in the mathematics classroom. After this laboratory unit, the course would follow its normal sequence, the models being used for classroom demonstrations as the corresponding theorems were studied. Another possible procedure would be to give the preliminary instructions in class and then have the models completed as an out-of-school project.

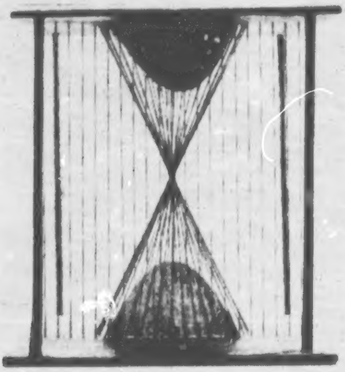


Figure 30

CARDBOARD MODELS

In the proof of the theorem, *There are five and only five regular convex polyhedrons,* the pupil is told that, because the sum of

LABORATORY SHEET NO. 1

I. PROJECT. *To construct a model which will represent correctly the geometrical relationships of the given theorem.*

II. THEOREM. *The plane passed through two diagonally opposite edges of a parallelepiped divides it into two equal triangular prisms.*

III. MATERIALS. 2 pieces of plywood, 12 in. by 12 in.
4 dowels, $\frac{3}{8}$ in. in diameter.
4 $\frac{3}{4}$-inch nails; 1 2-inch nail.
Glue.
Carpet warp.
Hammer.

IV. PROCEDURES.

A. Sketch in the space below the figure first as it appears in the text, second, as it would appear from a point 90° to the right, and third, as it would appear from a point 90° to the left.

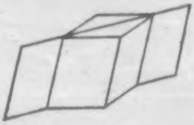


Figure 1

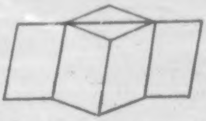


Figure 2

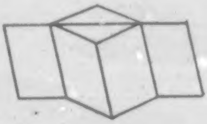


Figure 3

B. Draw below the configuration which appears in the lower plane and in a plane parallel to it.

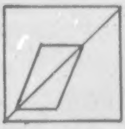


Figure 1

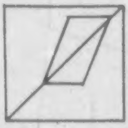


Figure 2

C. Reproduce the two drawings of B on the two pieces of plywood.

D. Using a two-inch nail, make holes in the plywood at one-half inch intervals to make the outline of the drawings of C.

E. On each piece of plywood locate four points at the corners, each one-half inch from the edge. Nail the four dowels to the plywood in these four positions. Place a drop of glue on the end of each dowel before driving the nail.

F. Lace carpet warp through the holes and from the upper to the lower piece of plywood to form the lateral faces of the figure. Use a different color of warp to represent planes which intersect the figure. If a large number of threads must pass through the same hole, that hole will necessarily have to be enlarged.

V. CONCLUSION. Make the necessary measurements and perform such calculations as will help you to verify that your model represents the correct relationships.

the face angles of a polyhedral angle must be less than 360°, he must place together the vertices of three, four, or five equilateral triangles, three squares, or three regular pentagons. The polyhedrons formed under these conditions are the tetrahedron, octahedron, icosahedron, hexahedron, and dodecahedron, respectively. No regular polyhedron is possible with regular polygons of more than five sides as faces, because the sum of the face angles at each vertex would be 360° or more. Here the lesson usually ends, but it can profitably be carried much further. Let the pupil build these polyhedrons with regular polygons cut from cardboard and fastened together with stamp hinges or gummed labels. For example, to build an octahedron, the pupil fastens together four equilateral triangles to form the first polyhedral angle and then adds one triangle at a time so that at each vertex he has a polyhedral angle with four 60-degree face angles; finally he has a polyhedron with eight faces.

An Archimedean or semi-regular polyhedron is a solid whose polyhedral angles are equal and whose faces are composed of two or more different kinds of regular polygons. The pupil can "discover" these solids by experimenting with equilateral triangles, squares, regular pentagons, and other figures to compute the number needed for a vertex and then completing the solid figure. The various possibilities are given in the table which follows:

ARCHIMEDEAN POLYHEDRONS

A. Polyhedrons with two kinds of faces and trihedral angles.

Units of Each Polyhedral Angle	*Number of Faces*	*Name of Polyhedron*
1. Two 120° angles, one 60° angle.	Four regular hexagons + four equilateral triangles.	Truncated tetrahedron
2. Two 135° angles, one 60° angle.	Six regular octagons + eight equilateral triangles.	Truncated cube
3. Two 120° angles, one 90° angle.	Eight regular hexagons + six squares.	Truncated octahedron
4. Two 144° angles, one 60° angle.	Twelve regular decagons + twenty equilateral triangles.	Truncated dodecahedron
5. Two 120° angles, one 108° angle.	Twelve regular pentagons + twenty regular hexagons.	Truncated icosahedron
6. Two 90° angles, one 60° angle.	Three squares + two equilateral triangles.	Archimedean prism

B. Polyhedrons with two kinds of faces and tetrahedral angles.

7. Two 90° angles, two 60° angles.	Six squares + eight equilateral triangles.	Cuboctahedron
8. Three 90° angles, one 60° angle.	Twelve squares + eight equilateral triangles.	Small rhombicuboctahedron
9. Two 108° angles, two 60° angles.	Twelve pentagons + twenty equilateral triangles.	Icosidodecahedron
10. Three 60° angles, one $\frac{180^\circ (n-2)}{n}$ angle.	Two regular polygons with n sides and $2n$ equilateral triangles.	Archimedean prismatoid

C. Polyhedrons with two kinds of faces and pentahedral angles.

11. One 90° angle, four 60° angles.	Six squares + thirty-two equilateral triangles.	Snub cube
12. One 108° angle, four 60° angles.	Twelve regular pentagons + eighty equilateral triangles.	Snub dodecahedron

D. Polyhedrons with three kinds of faces and trihedral angles.

13. One 90° angle, one 120° angle, one 135° angle.	Six regular octagons + eight regular hexagons + twelve squares.	Great rhombicuboctahedron
14. One 90° angle, one 120° angle, one 144° angle.	Twelve regular decagons + twenty regular hexagons + thirty squares.	Great rhombicosidodecahedron
15. One 108° angle, one 90° angle, one 60° angle.	Twelve regular pentagons + thirty squares + twenty equilateral triangles.	Small rhombicosidodecahedron

Let us suppose that a pupil elects to build a polyhedron whose faces are equilateral triangles and squares. According to the table, this will be a cuboctahedron, a small rhombicuboctahedron, or a snub cube. The pupil cuts from cardboard a number of equilateral triangles and squares, all with sides of the same length. Next, using stamp hinges, he fastens a cardboard square to each side of a cardboard triangle. Now he may place either a square or a triangle to each vertex of the original triangle and in the space between the squares. If he chooses squares, he is on the way to building a small rhombicuboctahedron. He continues in the same manner, remembering that each vertex is a tetrahedral angle composed of three 90-degree angles and one 60-degree angle.

When the polyhedron is completed, the pupil may cut it apart, marking each pair of adjoining sides "plus" and "minus" as he does so. Ultimately, the polyhedron can be spread out as in Figure 31. The pupil now has a pattern which can be used in making a solid cardboard model. The first step is to construct accurately the pattern on cardboard, using ruler and compass. White Rockton Index (110-lb. basis) is a satisfactory grade of cardboard to use for this purpose. Lapels are drawn on the edges marked "minus." The second step is to cut out the pattern and to fold it on each common side of adjacent polygons. The third step is to glue the figure together, obtaining, in this example, a small rhombicuboctahedron (Fig. 32). Household cement can be used as an adhesive.

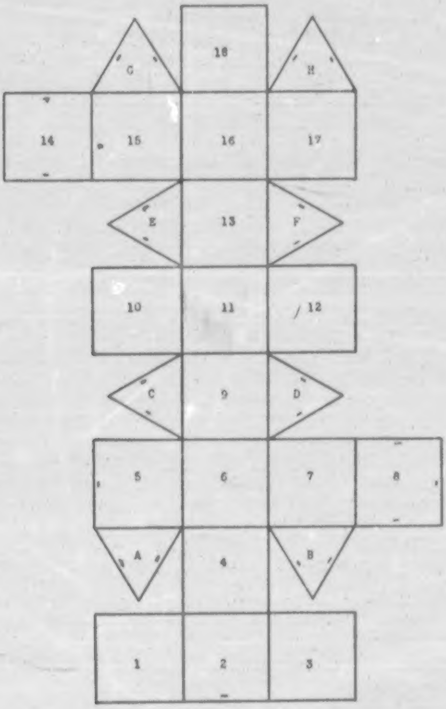


Figure 31

Variations of this polyhedron may be obtained by making pyramids with regular triangular or quadrangular bases and gluing them to the faces of the cuboctahedron. However, better-looking solids are formed if the figure is built up with equilateral tri-

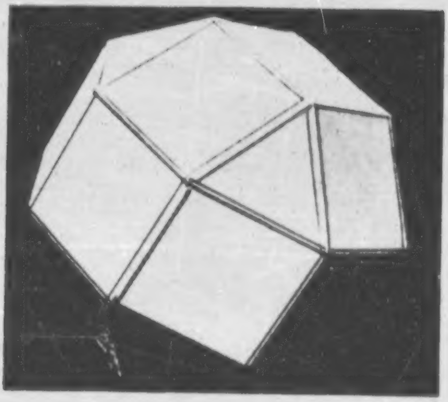


Figure 32

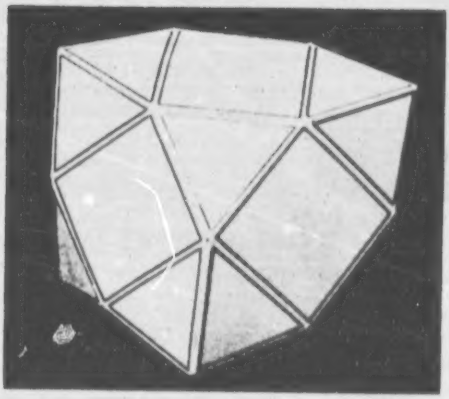


Figure 33

angles, isosceles triangles, and squares in the manner just described. Figure 34 represents an original pattern created by a student who saw that it was unnecessary to build the solid and that alternate squares in the pattern could be replaced by four isosceles triangles. By studying the numbers which represent corresponding parts of the two solids (Figs. 31 and 34), the reader will be able to discover how the process was carried out. Figure 33 shows the completed model. In a similar manner one can obtain a star-like polyhedron by replacing each square with four isosceles triangles and each equilateral triangle with three isosceles triangles. Many unusual concave polyhedrons can be discovered by using semi-regular polyhedrons as a basis and replacing faces with isosceles triangles of different altitudes.

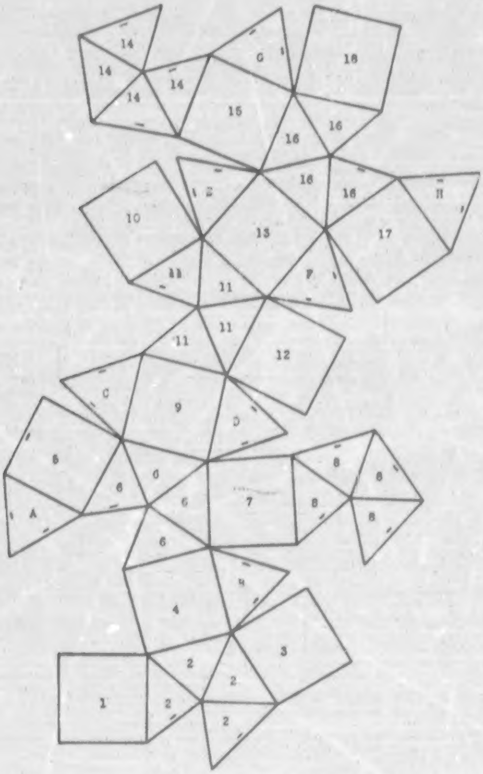

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Figure 34

Interesting exercises might be: What is the length of each edge of regular (or semi-regular) polyhedrons so that they will have equal volumes? Equal areas? How many axes of symmetry does each polyhedron have? Obtain the semi-regular solids by cutting off the vertices of the regular solids. For the semi-regular polyhedrons, verify Euler's theorem that *the number of faces of any convex polyhedron, together with the number of vertices, is two more than the number of edges.*

CELLULOID MODELS

It should be stated in advance that celluloid is not easy to handle and that a certain amount of practice in using it is necessary before satisfactory results can be obtained. Acetone (or nail polish remover which has an acetone base) is recommended as a substitute for an adhesive; Scotch tape can also be used to fasten edges together. Lapels, which are necessary for cardboard models, are optional in the case of celluloid models.

The abstract situation of inscribed and circumscribed spheres is very hard to visualize and frequently results in the meaningless memorization of the definition by the pupils. To add reality to the representation of inscribed spheres, celluloid can be employed effectively to make models in which polyhedrons, cones, or cylinders circumscribe spheres. Any kind of ball (preferably colored)—ping pong ball, tennis ball, celluloid ball—can be used to represent a sphere. The following formulas give the relationships between the diameter of the inscribed sphere and an edge or element of the circumscribed solid.

Solid	*Diameter of Inscribed Sphere*
Cube, edge a	a
Tetrahedron, edge a	$\frac{a}{6}\sqrt{6}$
Octahedron, edge a	$\frac{a}{3}\sqrt{6}$
Icosahedron, edge a	$\frac{a}{6}(3 + \sqrt{5})\sqrt{3}$
Dodecahedron, edge a	$a\sqrt{\frac{25 + 11\sqrt{5}}{10}}$
Cylinder, radius r altitude $2r$	$2r$
Cone, radius r altitude $r\sqrt{3}$	$\frac{2}{3}r\sqrt{3}$

The superior pupils can be led to discover these relationships and the average pupils can verify their correctness.

Laboratory Sheet No. 2 gives complete directions for making these celluloid models.

LABORATORY SHEET NO. 2

I. PROJECT. *To construct a model which will represent a ——————— circumscribed about a sphere.*

II. MATERIALS: Celluloid
Acetone
Ball
Calipers

III. PROCEDURES.

A. Preparation.
1. The formula for the diameter (d) of a sphere inscribed in a ——————— of edge a is ———————.
2. Solve this formula for a.
3. Calculate the numerical coefficient of d: $a =$ ——— d.

B. Measure the diameter of the ball.
First measurement: ———————.
Second measurement: ———————.
Third measurement: ———————.
Average diameter: ———————.

C. Using the average diameter of B and the formula of A3, calculate the length of an edge of the required polyhedron.
$a =$ ———.

D. Construct a cardboard polyhedron having each edge the length calculated in C. Place the ball inside the polyhedron before the figure is completely closed.
Does the polyhedron circumscribe the sphere? If it does not, check your work in A, B, and C.

E. Repeat D, using celluloid instead of cardboard.

IV. CONCLUSION.

A. When is a polyhedron said to be circumscribed about a sphere?

B. Does your celluloid polyhedron meet these requirements; that is, is it circumscribed about the sphere?

CONCLUSION

This laboratory approach to solid geometry has been so satisfactory that it has become an integral part of our course of study. It has been responsible for the high degree of achievement of several important aims of the course, namely: to acquire an understanding of the spatial relationships which exist in the world about us, to become acquainted with spatial forms, to train the spatial imagination, and to obtain an analytical insight into our environment. For the student, it has aroused a vital interest in the subject of solid geometry, and for the teacher, it has presented a challenge to seek and to test new devices of a multi-sensory character which may be profitably incorporated into an ever-growing course of study.

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GEOMETRIC DRAWING

H. V. Baravalle

GEOMETRIC DRAWING is a mathematical laboratory method to be compared with the laboratory work done in the natural sciences. It stimulates mathematical interest and serves as a basis for other mathematics courses. Besides the opportunities which it offers in connection with mathematics, geometric drawing helps to develop manual skill. Many techniques are taught in the course. There is the handling of compasses, the drawing with triangles, the use of ruling pens and black ink, the application of color to both lines and areas, including shadow effects on solids and curved surfaces, and, finally, lettering. Through these techniques such artistic qualities as a sense of proportion when placing figures and script into a given space and skill in combining colors are developed. In referring to the script, it often happens that the handwriting of students improves as they develop their sense of form and proportion and skill in the use of their hands.

Geometric drawing thus holds a middle position between the academic work in the school and the arts and crafts, and offers special opportunities within the general educational tasks. It can be applied in various forms to different school levels. Courses in geometric drawing have been given since 1920 in the Waldorf School, both in the Teachers Training College and throughout the Junior and Senior High School, and later in about a dozen European schools. In this country pioneer work with geometric drawing has been done, for the high school level, mainly at the Edgewood School in Greenwich, Conn., and the High-Mowing School in Wilton, N. H., and for the college level at Adelphi College in Garden City, N. Y.

An introduction of geometric drawing in the *seventh grade* can be based on the natural sense of form and regularity which is inherent in the students, both boys and girls, at this age. The appreciation of the beauty in geometric designs predominates at this age over interest in scientific problems. Therefore the approach will start with the regular forms.

The course introduces equilateral triangles, squares, and regular polygons. The construction of regular polygons is carried out by dividing a circle; therefore, among the first steps are the exercises for dividing a circle into a given number of equal parts. The most natural division of a circle is the division into six arcs where the circle's own radius is used as measure to cut it. If one starts with a vertical and a horizontal diameter of a circle and then cuts the circle with the radius from any one of the four end points of the diameters to both sides, a circle of twelve equal arcs is obtained. Afterwards straight lines can be drawn between the points of division. In Figure 1 a twelve-sided star polygon is drawn by connecting every one of the twelve points on the circle with the fifth one following it. The spaces between the various sections of the star polygon lines are done alternately in black and white, but the students often use either water color or crayon to differentiate them.

The face of a compass can be drawn as a practical application of a star polygon. This requires a division into sixteen parts instead of twelve. The division of a circle into sixteen parts introduces the construction of bisecting arches, for which either compasses or strips of paper can be used. Starting with the vertical and horizontal diameters of a circle, the bisection is first carried out with the four quarters of the circle making eight parts, and the second bisection results in sixteen parts. Afterwards every point is connected with every seventh following it around the circle. A sixteen-sided star polygon is thus obtained. Finally the filling in of the black areas accentuates the four main regions, North, South, East, and West, and the angles of forty-five degrees between them as shown in Figure 2.

If the twenty-four points of division around a circle are joined by all the lines that can be drawn between them, a geometric figure is produced as shown in Figure 3 within a twenty-four-sided polygon. A number of drawings of this kind, up to sixty-sided polygons, with all diagonals, have repeatedly been made by students in their free time.

Many other exercises can be carried out with regular figures.

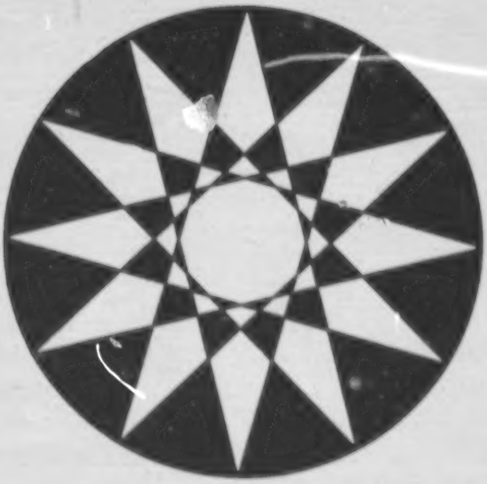

Figure 1. Twelve-sided Star Polygon

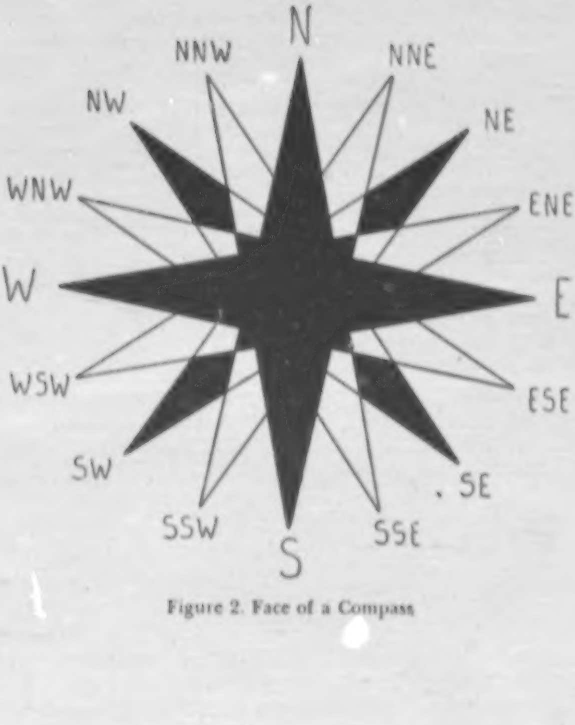
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Figure 2. Face of a Compass

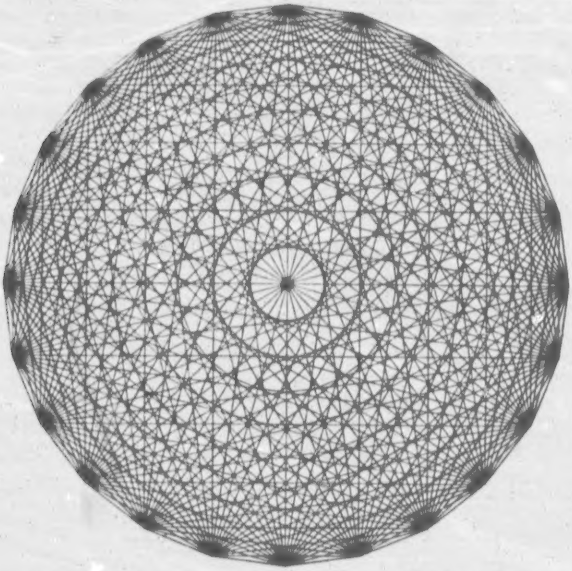


Figure 3. Twenty-four-sided Polygon with All Diagonals

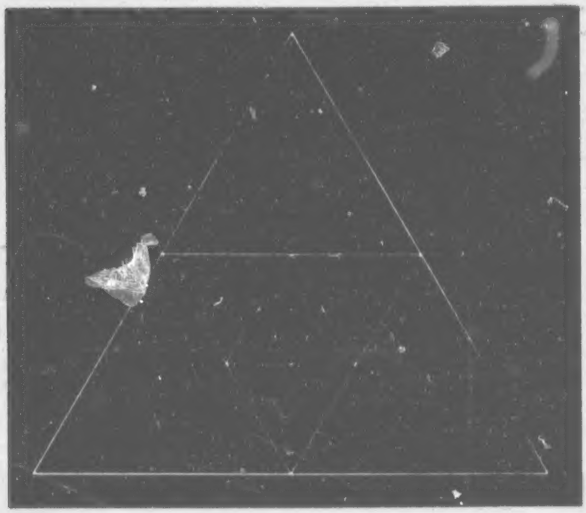


Figure 4. Series of Triangles Demonstrating the Corresponding Proportions between Sides and Areas

An example is given in Figure 4. It is begun with the outside equilateral triangle. In this case the various ways of drawing equilateral triangles will be shown, one starting with the division of a circle into six equal parts, and then connecting every second point and skipping the intermediate points. A second way begins with the base of the triangle and determines its top through the intersection of two circular arches that have their centers in the end points of the base and their radii equaling the length of the base. A third construction uses a drawing-triangle with an angle of sixty degrees.

After the outside equilateral triangle has been completed, each side is bisected and the mid-points are mutually connected. These connecting lines form a smaller triangle inside the first one in reverse position. Its sides are equal to one-half the sides of the outside triangle, while its area is one-quarter that of the first triangle. The process is then repeated with the inner triangle, and thus a third triangle is formed inside the second one. Its sides are ½ of ½ = ¼ of the first, and its area is ¼ of ¼ = 1/16 of the first.

This succession of triangles can be carried further by always drawing another triangle inside the last one. These triangles stand alternately upright and reversed and represent a geometric progression with the ratio ½ as to the sides and ¼ as to the areas. It is characteristic that even in algebra, progressions of numbers with constant ratios are named geometric progressions. Drawing of the kind described will give the student understanding of this term.

A further example of a geometric progression is given in Figure 5, in which twelve squares of the same size are arranged in the form of a ring. The corners at which two successive squares of the same ring touch each other lie at equal intervals along a circle. The drawing of the figure is started with this circle, which is divided into twenty-four parts. At every second one of these twenty-four points two adjoining squares come together. Through the other intermediate twelve points radii are drawn and on them lie the other vertices of the squares pointing outward and toward the center. These vertices are determined on their radii with the help of diagonals. First, in each square there is drawn the diagonal con-

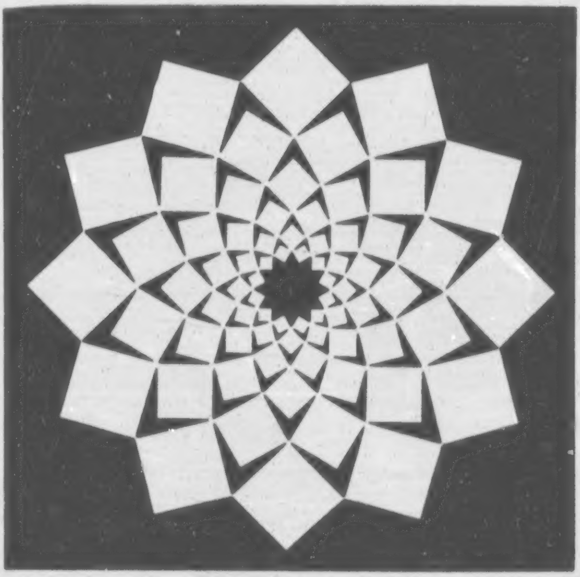


Figure 5. Geometric Progression

necting the two points lying on the circle. Its mid-point is its intersection with a radius. By measuring the length of half the diagonal from the mid-point outward and toward the center, one obtains the outside and inside vertices of the squares. Between the inner vertices lies the next ring of squares. The circle which can be drawn through these points plays the same part for the second ring as the original circle does for the outer ring of squares. Thus one ring of squares after another can be added.

When the drawing is completed, a number of geometric facts can be discovered. If one imagines, for instance, the sides of any square in any ring prolonged, he sees that they lead into the sides of two other squares of the same ring. The extensions of the sides of these two squares also contain the sides of a fourth square. These four squares form the corners of a larger square. Thus every ring of squares shows itself composed of 3 × 4 squares. These squares form a twelve-sided star polygon. Furthermore, if one follows any side of a square from the outer corner to the inner and then continues the movement along the adjoining side of the next square, he makes the successive corners reached in this way form a logarithmic spiral. There are twenty-four logarithmic spirals in the drawing. The students will also discover that the pattern of this geometric progression can be found in nature. A geometric pattern of this kind is displayed by the middle part of a sunflower or, on a smaller scale, by daisies; logarithmic spirals are to be found in pine cones, in shells, and the like.

Exercises of this kind will demonstrate that studies based on geometric principles lead to manifestations which nature also produces. The student will no longer consider mathematics isolated from nature, but as a part of it.

In the *eighth grade* the student's mind tends to be more realistic. At this stage of his development it is therefore particularly valuable for him to experience certain practical sides of geometry. Perspective drawing can be successfully introduced at this age. This study often leads a kind of foster-child existence in school curricula because perspective is generally affiliated with the arts, although it is essentially a geometric subject. It starts with basic constructions and opens possibilities for a variety of applications. The perspective series is an example. Its construction solves the problem of how to obtain perspective drawings of objects placed at equal intervals and leading from the foreground into the distance. The diminishing intervals illustrate perspective diminutions, and the task of the construction is to apply these diminutions correctly. The construction is based upon three facts: First, in perspective pictures straight lines appear as straight lines. Second, distant objects are on a reduced scale. By combining these two facts the vanishing points are obtained, i.e., the points to which parallel lines converge. The third is the fact that in perspective drawings that are in a normal position vertical lines remain vertical.

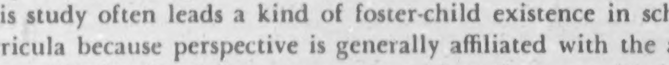


Figure 6 and Figure 7. Construction of Perspective Series

The construction of perspective series is shown in Figures 6 and 7. In Figure 6 is drawn a succession of equidistant vertical lines. Three horizontal lines which connect their tops, their bases, and their mid-points are added. Figure 7 shows the perspective drawing of the same combination of lines. First, the two foremost verticals can be chosen at will, the smaller one being considered farther away from the spectator. The straight lines connecting the tops and the bases of the verticals converge to a vanishing point and so does the third horizontal line connecting the mid-points. Then the first diagonal can be added by connecting the top of the first vertical with the mid-point of the second. Where this diagonal reaches the line connecting the bases, there is the base of the third vertical, which is determined by this intersection and can be drawn in the picture. Now the second diagonal can be added by connecting the top of the second vertical with the middle of the third one, and its intersection with the base line furnishes the base of the fourth vertical. Thus the construction can be continued.

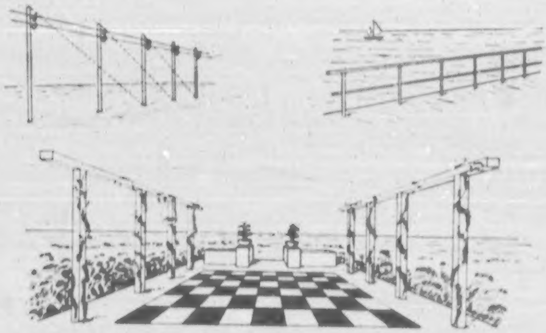


Figure 8, Figure 9, and Figure 10. Applications of Perspective Series

The construction of perspective series finds its application in many drawings. It can also be applied to horizontal lines. Two examples of vertical lines and one example of horizontal lines are shown in Figures 8, 9, and 10.

The preceding exercises use only one vanishing point. In the following examples a second vanishing point is added. The vanishing points of horizontal lines all lie on the same "perspective horizon." A whole city can be drawn on this principle. Figure 11 shows the perspective view of a city, and the drawing is composed merely of verticals and two sets of horizontals, all converging, either to the right or to the left, to two vanishing points on the perspective horizon.

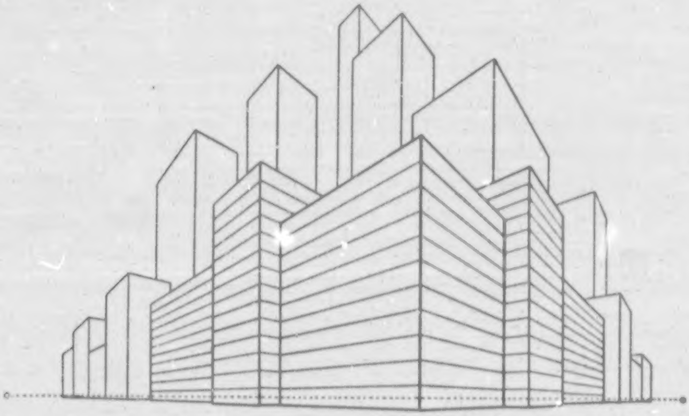


Figure 11. Perspective View Constructed with Two Vanishing Points

A next step leads to the construction of roofs, an exercise which introduces inclined planes. There are mainly two kinds of roofs: the pointed roof and the gable roof. The perspective construction of a pointed roof is shown in Figure 12. It is carried out with the help of diagonals drawn across the base of the roof. For this purpose the invisible part of the house has to be added, as indicated

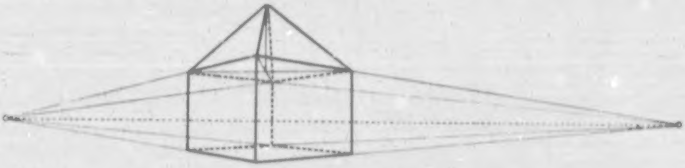


Figure 12. Construction of a Pointed Roof

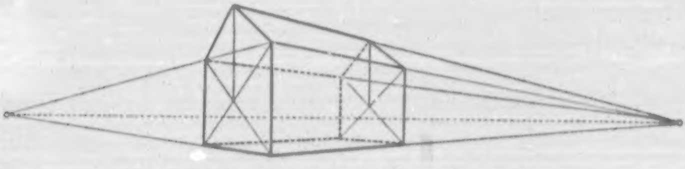


Figure 13. Construction of a Gable Roof

into contact with mathematical facts. A subject which is particularly well adapted for this purpose is the study of curves.

There are two ways of drawing curves, either through successions of points or by means of tangents. Tangents offer greater advantages for the beginner, because they form the curves directly without the drawing of a curved line being necessary. Examples for the study of curves through their tangents are given in the following figures. The first shows the construction of a parabola (Fig. 15). The figure starts with the two outer lines, each of which is divided into equal lengths. The points of division are then connected through straight lines. The highest point on the left line is connected with the lowest point on the right line. Then the next highest point on the left is connected with the next lowest on the right. The successions proceed on both lines in opposite directions on one line downwards, on the other upwards. The areas between the sections of these connecting lines are completed in a checkerboard manner alternately in black and white. Following either the white or the black areas, one obtains further parabolas besides the one formed by the tangents.

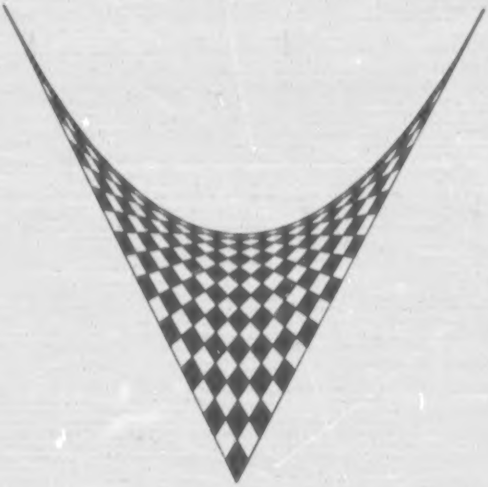


Figure 15. Construction of a Parabola

A stimulating experience in the realm of geometry is to apply in the figure by means of dotted lines. The peak of the roof lies vertically above the point of intersection of the diagonals.

The construction of a gable roof is shown in Figure 13. The gable roof rises to a ridge. It connects in different ways the four walls of the building. Two of them rise up to gables, whereas the other two end in horizontal lines. The peaks of the gables are vertically above the middle points of their walls. Their construction is therefore again carried out with the help of diagonals which are drawn across the gable walls. The peaks of the gables in the perspective drawing lie vertically above the intersection points of the diagonals.

Finally, the study of perspective can be extended to shadow constructions. An example is shown in Figure 14. After the drawing of the gate has been completed and the position of the sun chosen, the construction proceeds with two groups of lines. First, there are lines connecting various points of the gate with the position of the sun; second, there are connecting lines drawn toward the ground plan of the sun which lies vertically below the sun on the perspective horizon shown by dotted lines in the drawing. The intersections of these dotted lines with those drawn to the sun furnish the shadow points.

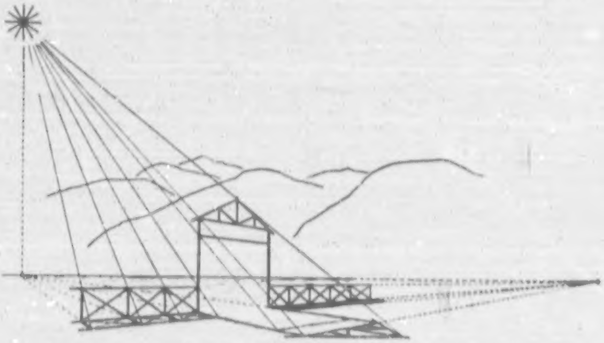


Figure 14. Shadow Construction in Perspective

In the *ninth grade,* when the mental activity of the student begins to take on a directly scientific trend, the point of view for the geometric drawing exercises shifts from the realistic to the primarily mathematical angle. The student will then be brought different ways to construct the same curve. In Figure 16 parabolas are obtained through the intersection of two series of lines. The first series consists of parallel horizontal lines following one another at equal distances. The second series consists of concentric circles increasing their radii in equal steps. The areas between the various sections of the lines are again done alternately in black and white. Following either the black or the white areas leads to parabolas. The drawing thus brings forth not only one parabola but a complete group of confocal parabolas turned upwards and downwards.

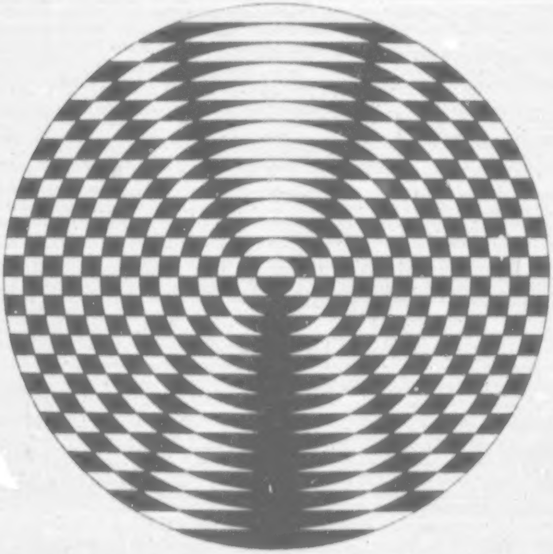


Figure 16. Parabolas Constructed with Straight Lines and Circles

A third construction of parabolas follows the method which Galileo used in his investigations of the laws of free falling bodies. Galileo's two laws are that the distances covered from the beginning of the fall increase in the ratio of square numbers and that the distances covered in successive seconds have the ratio of odd numbers. These ratios determine in Figure 17 the distances between the horizontal lines. These follow one another in intervals of 1; 3; 5; 7; 9 . . . units, and in heights of 1; 4; 9; 16; 25 . . . from the top line. The vertical lines are drawn at equal distances from each other. The narrow inner parabolas take one vertical step combined with one horizontal step. The next wider parabola combines each vertical step with two horizontal steps. The widest outside parabolas take with every vertical step three steps sideways. The drawing could be continued in the same way.

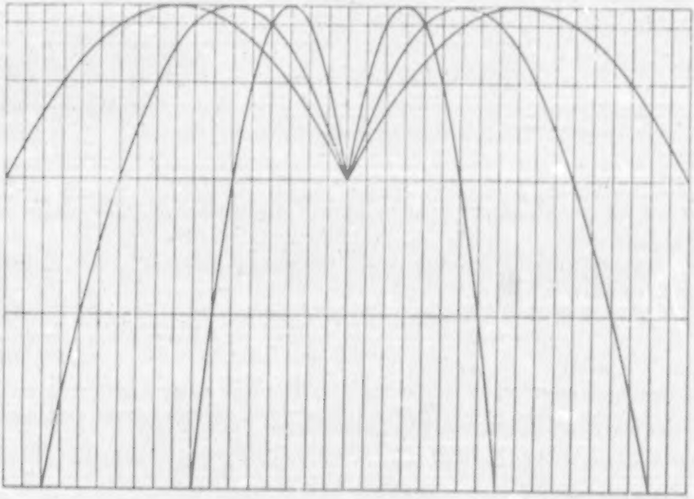


Figure 17. Parabolas Constructed on the Ratios of Distances

Further exercises show altered conditions in various constructions which bring about certain changes in the curves. Instead of using straight lines and circles as in Figure 16, the next figure (Fig. 18) is composed of two groups of concentric circles. The radii increase in both groups at an equal rate. The succession of either adjoining black or adjoining white areas leads in one direction to ellipses and in the other to hyperbolas. The totality of these ellipses and hyperbolas represents a group of confocal curves.

Exercises of this kind are preparation for further mathematical concepts. Having drawn the system of confocal ellipses and hyperbolas, a student will easily comprehend later that ellipses and

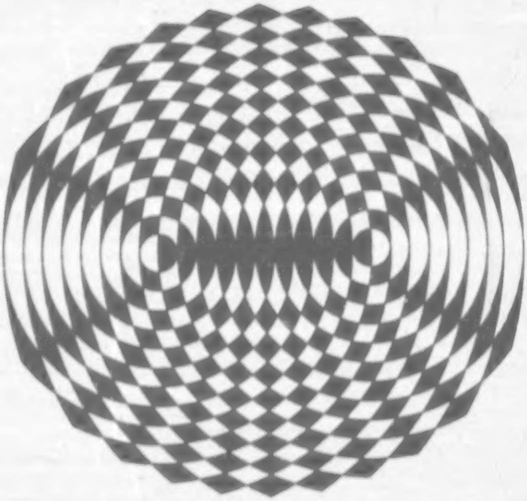


Figure 18. Confocal Ellipses and Hyperbolas

hyperbolas, both being curves of the second order, have equations in this position which are different from one another merely through a single plus sign changing into a minus sign. Another construction which displays polarity between ellipses and hyperbolas is shown in Figures 19 and 20.

The constructions in both these figures start with a circle and a given point. The only difference between them is in the fact that the given point in the first case lies inside the circle and, in the second, outside. Straight lines are drawn radiating from the point chosen, and where these lines intersect the circle, perpendiculars are erected upon them. These perpendicular lines form the tangents in the case of the ellipse and in the other of the hyperbola.

The more aspects one can gain of the same curve, the more interest and vitality will the study of geometry produce. A third way of constructing hyperbolas is shown in Figure 21. This construction is deduced from the projection of a hyperboloid and thus leads on from plane to solid geometry.

To begin with, the circle is divided into twenty-four equal parts. By drawing vertical connecting lines between these points, a scale composed of larger and smaller sections is obtained on the

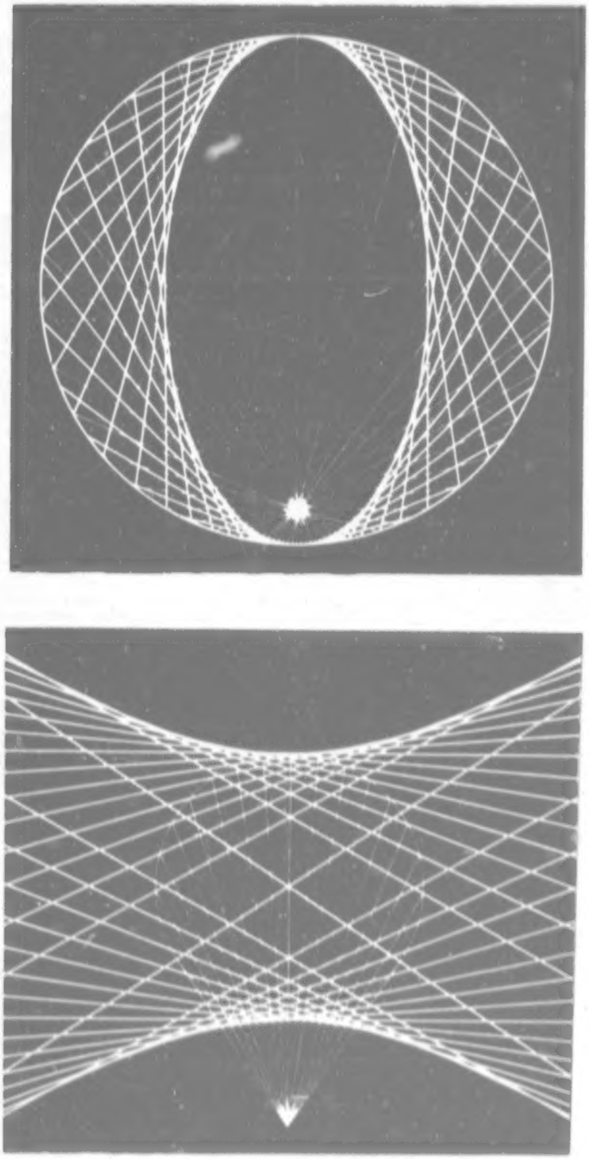
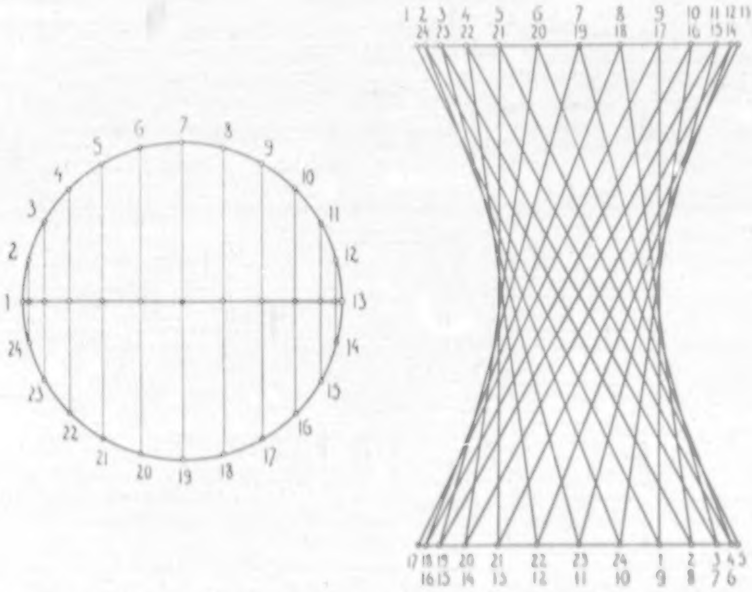


Figure 19 and Figure 20. Construction of Ellipses and Hyperbolas horizontal diameter. This scale is then transferred to both horizontal lines of Figure 22. The points of the scale are numbered according to the successive points around the circle. The numbers of the scale are directly transferred to the top line in Figure 22.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17 18 19 20 21 22 23 24 1 2 3 4 5
16 15 14 13 12 11 10 9 8 7 6

Figure 21 and Figure 22. Construction of a Hyperbola from Harmonious Motion

The points on the base scale also show numbers, but they are transposed. In the base scale number 1 lies under number 9 of the top scale, 2 under 10, 3 under 11, 4 under 12, and so on. Then the points with the same numbers are connected. These connecting lines form the tangents of a hyperbola. A variety of different forms of hyperbolas can be obtained by different transpositions between the two scales.

Further drawings can be made to demonstrate different theorems. An example is shown in Figure 23. It demonstrates in a succession of five steps the theorem of Pythagoras.

In the first instance (top left) the square of the hypotenuse is marked in black. In the second (top right) a triangle is added

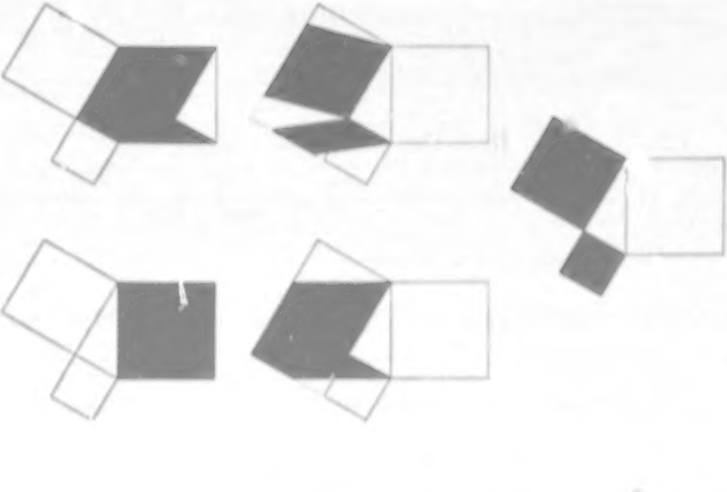

Figure 23. Theorem of Pythagoras

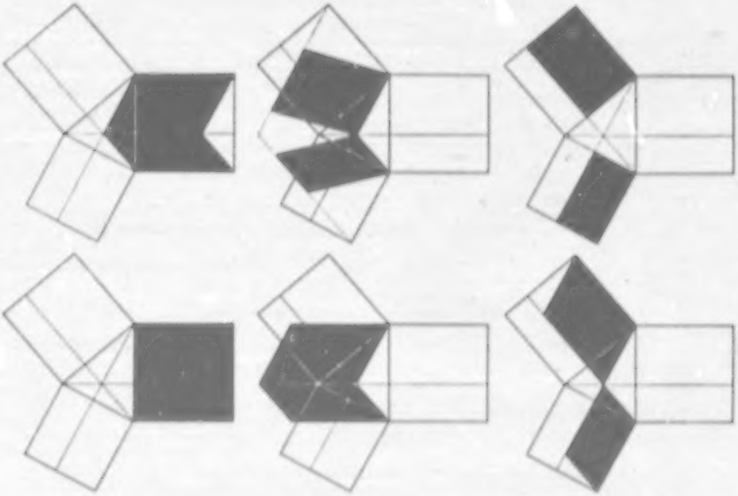

Figure 24. Law of Cosines

on top of the black square and then an equal triangle is omitted at its base. Between the second and the third stage (middle left) the form of the black area does not change; it is just moved upward until its base line is lifted to the height of its top. In the fourth step (middle right) the black area is split. Each of the two parts takes on the form of a parallelogram and is moved sideways with its bases and heights left unchanged; therefore it preserves its area. This motion can be continued until the black areas reach their last phase and become identical with the squares of the legs. This completes the derivation of the theorem of Pythagoras. It not only demonstrates that the area of the square on the hypotenuse equals the sum of the areas of the squares of the legs, but it shows the actual transformation. The same derivation can be extended from the right triangle to the oblique triangle. In this form it demonstrates the cosine law of trigonometry (Fig. 24).

The further plans for geometric drawing continue up to the senior grade of high school and include an introduction of the descriptive geometry of solids with both plane and curved surfaces. The methods used remain the same. The examples elaborate especially on those facts which display geometric contents in their most complete form, for instance, the study of regular solids. The construction of geometric models by the student can complete this approach. In the last years more specialized work which includes problems of physics (mechanics and optics), of projective geometry, and of architectural drawing is carried out by the students.

CURVE-STITCHING IN GEOMETRY

Carol V. McCamman

While using the mathematical library of Dr. Sophia H. Levy of the University of California, I was attracted by a little book entitled *A Rhythmic Approach to Mathematics.*[1] Its numerous illustrations were designs, some simple, some intricate, many beautifully colored, but all composed of straight lines. I found that all of them had been produced by a simple plan of stitching on cards, and that many had been worked by very young children. I immediately made some of the designs myself and was fascinated by the possibilities of the method.

My enthusiasm for these designs was shared by my geometry classes, and also by many others who saw them. The first year I took about twenty minutes of a class period to describe the method to my students, and showed them several of the designs I had made. This was all that was required to interest some of them in making their own designs. These designs, shown to the class, in turn aroused the interest of others.

The next year, in order to have every student take part in the work, I spent a full class period in explaining the method and in analyzing several designs made by former students. Each student was asked to plan a design of his own, which he would later work out in colors of his own choice. Not only did most of the students have their plans the following day, but many had started stitching, and some brought in completed projects. Two class periods were long enough for everyone to get well started. Any further time needed was spent outside class. In many cases students made additional designs, some of the more enthusiastic members producing as many as six or eight.

The method used in making these designs is described in *A Rhythmic Approach to Mathematics.* This book, long out of print, was reproduced[2] in 1941 through the efforts and support of Mrs. W. F. Dummer, of Chicago, and interested teachers. In an introduction to the book, Mrs. Mary Everest Boole, the wife of the English mathematician George Boole, explains the way in which this method "of evoking the geometric instinct" was developed on the basis of the ideas of Boole and of the French mathematician Boulanger. The purposes of curve-stitching, as stated by Mrs. Somervell, are to provide the child with a background of pleasurable experiences before he begins the actual study of mathematics; to develop in him an awareness of the intimate relation between number, form, movement, and the process of thought; and to enable him to translate, by means of his sense of relation, any of these into terms of any other.

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[1] Somervell, Edith L. *A Rhythmic Approach to Mathematics.* George Philip & Son, Ltd., London, 1906.

When curve-stitching is done by students of geometry, both the methods and the aims are, of course, different from those for young children. However, the work in an important sense provides a background of experience for mathematics yet to be studied, particularly analytic geometry and projective geometry. To many beginning students of analytic geometry, the concept of an envelope is abstract and meaningless; curve-stitching consists, actually, in constructing such a curve by making stitches which are the tangents to it.

I shall indicate briefly how to make several of the basic curves. The parabola is obtained by connecting equally spaced points in opposite directions on the sides of an angle (Fig. 1). Stitches are made from A to A', from B' to B, from C to C', from D' to D, etc. In making this and similar curves, the following rules should be observed:

(a) From the first hole on the lower side of the angle put the needle into the first hole on the other end of the upper side of the angle.

(b) On the "wrong" side of the card, always put the needle into the next hole; put all knots and marks on this side.

(c) On the "right" side, put the needle into the hole next to the beginning of the last long stitch.

Many interesting designs are based on circles. For example, draw two concentric circles and prick them so that the number of holes on the larger circle is twice the number on the smaller circle (Fig. 2). Then stitch from *A* to *A′*, *B′* to *B*, *C* to *C′*, *D′* to *D*, etc.,

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[2] Planographed copies are available at fifty cents each from Miss L. E. Christman, 1217 Elmdale Avenue, Chicago, Illinois.

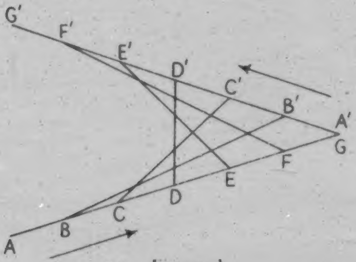


Figure 1

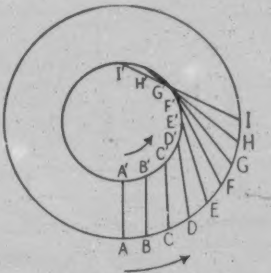


Figure 2

continuing to connect successive points until every hole on the outer circle has been used. To do this it will be necessary to go around the small circle twice. The curve obtained can be varied by changing the ratio of the numbers of points on the two circles, and also by stitching clockwise on one circle and counterclockwise on the other.

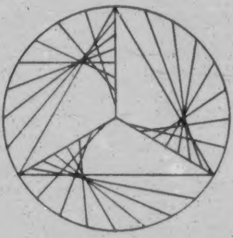


Figure 3

An interesting design is obtained by working one-third of a circle against its radius (Fig. 3).

An almost endless variety of designs can be produced. Young children, especially, will be interested in the "curve of pursuit" (suggested by Mrs. Somervell), in which the stitches show the successive intentions of a dog as he chases a rabbit, while the curve obtained represents his path.

Students using this method should be encouraged to originate their own designs and color schemes. As the author states, "Verbal explanations of how to work the designs represented would have a quite false appearance of being very complicated." It is relatively difficult to copy designs others have made, but practice with needle and thread for an hour or so will enable anyone to start making his own designs.

When I first became interested in these designs, I wondered whether, in geometry classes composed largely of boys, the work would be stigmatized as "embroidery" and "sissy." I need not have worried. Not once did anyone seriously suggest that curve-stitching was a feminine occupation. On the whole, the boys seemed rather more interested than the girls—perhaps because of the novelty—and produced designs which were more original and, in most cases, better executed than those of the girls.

When the completed designs were exhibited, the students voted for their favorites. One of the best liked was a pale pink and blue design, very carefully worked, made by one of the stalwarts of the football team. An intricate and lacy design was made by a boy noisy in voice and manner, who was so unfamiliar with sewing that he thought the needle had to be tied to the end of the thread. A particularly striking design was made by a Chinese boy who at one time had been considered incapable of taking the regular mathematics courses. His chief difficulty was his inability to express himself in a strange language. His design, carried out in bright colors of fine silk thread, showed excellent workmanship and originality.

I have found curve-stitching a valuable supplement to the regular work in geometry. It requires only a small amount of class time. It makes the students aware of geometric relationships and develops their interest in the geometric basis of design. They seem to find real satisfaction in planning and making the designs. The interested and critical appraisal which the class makes of each new design is an incentive to good work, as is also the favorable comment the designs receive when they are exhibited in the school hall. Some students who have not been doing well in geometry find in this work a new opportunity to be successful and to earn the praise of their classmates. In many such cases, the increased interest in geometry seemed to carry over to subsequent work.

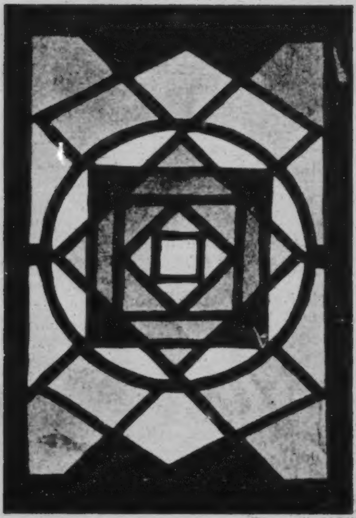
WINDOW TRANSPARENCIES

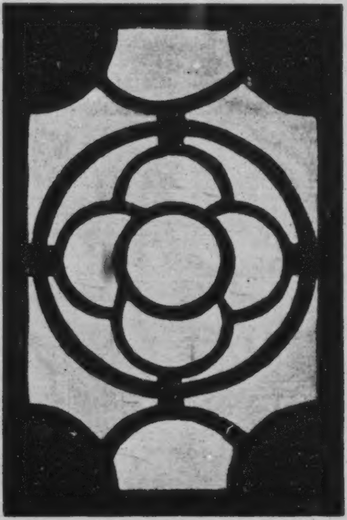
J. Anna Tennyson

Pupils enjoy making transparencies that simulate stained glass windows. This project may be the culmination of the construction work in intuitive geometry.

A small preliminary design is made on white paper. All lines are doubled, as most of the paper is to be cut away and only the narrow strips that outline the design are to be left. Black cover paper, not too stiff, is cut to fit the windowpane to be covered. The original design, enlarged to proper proportions, is constructed on the black paper. White or light colored pencils make construction lines that are most easily seen. The paper between the narrow strips that represent the leaded seams in a stained glass window is cut away by means of a razor blade or small, sharp scissors.

There is now left a skeleton-like design. Thin paper of various colors or colored cellophane is used to replace the cut-out paper.

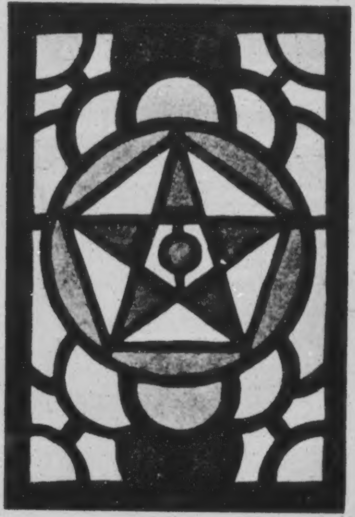


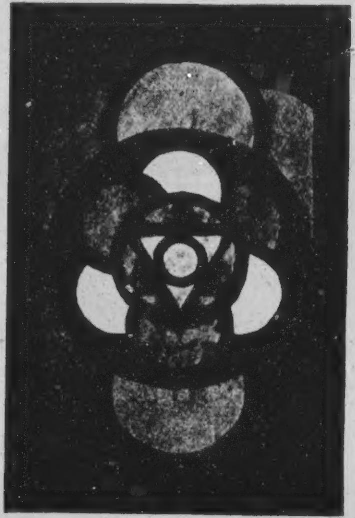


To heighten the resemblance to stained glass, the paper may be oiled or dipped in melted paraffin before it is used.

The finished transparencies are suggestive of mathematical forms and figures and create an atmosphere in which pupils are better able to enjoy and appreciate the role of mathematics in the art of design.

At Christmas stars and geometric Christmas trees used as basic figures in the design add a festive air to the classroom windows.





MATHEMATICAL DEMONSTRATIONS AND EXHIBITS

Phillip S. Jones

THE VARIETY of interested and appreciative comments elicited from both student and adult spectators at the annual demonstration-exhibits prepared for several years past by the mathematics department of The Edison Institute of Technology has led to the conclusion that more should be done about this method of increasing interest in and appreciation of mathematics. Scattered periodical articles[1] have told of occasional mathematical exhibits and the fun and pedagogical values derived from them. However, repeated expressions of surprise at the fact that mathematics had anything to exhibit outside of textbooks or a variety of apparatus ready to supplement blackboard demonstrations proved that the advice to "dramatize the role of mathematics in modern life" [2] was not only excellent and deserving of more attention, but needed to be carried out in more concrete ways than are used at present. It is the aim of this account of portions of past exhibits and of the article on "Mathematical Apparatus" (pages 212-225) to focus more attention upon the values of dramatization and to offer a few practical suggestions which may help teachers in starting such a program.

These are the principles which have guided the construction and arrangement of our exhibits. (a) An exhibit must gain and retain the interest of the viewer. (b) Hence, it must be geared to the level of an intelligent but non-mathematically trained student or adult. (c) It must not so cater to popular interest and lack of training that it makes mathematics seem trivial or a collection of puzzles, recreations, classical solved and unsolved problems and paradoxes. (d) It must sustain interest in mathematics by pointing out that the subject is alive and "on the hoof" where it may be seen in daily life by any observer, and by showing its more obscure applications and relationships. (e) As a school demonstration, it should show largely the actual materials of daily class instruction, and, in this connection, should use the instruments and gadgets on hand for classroom instruction. (f) Accompanied by diagrams and direct explanation, it should demonstrate relationships conclusively and tell an interesting story where a similar but static display is ineffectual.

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[1] Reference numbers 1 to 32 refer to notes at the end of this article.

With this last point in mind, each of the sections of these exhibits has consisted of a table display of apparatus or models behind which stood a student to explain the mathematics of the section by using the apparatus and models plus a set of drawings and pictures tacked to a display board behind him. Thus each part of the demonstration was carried out tangibly by models, many of them movable, audibly by the students' discussion, and visually through diagrams. The organization of one exhibit was as follows.

SECTION 1. THE CONIC SECTIONS AND THE ELLIPSE (FIG. 1)

On the table. String models of the conic sections and ruled quadric surfaces (these were made in the school machine shop,

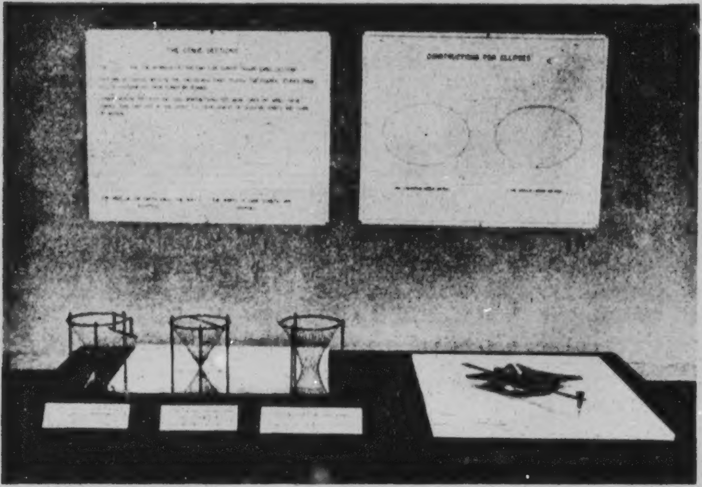

Figure 1. The Conic Sections and the Ellipse

but could easily be made from cereal boxes); drawing board and paper with two thumbtacks stuck in and a loose string tied around them ready to demonstrate the pin and string construction of an ellipse; an elliptic trammel or ellipsograph.[3]

Pictorial display. Poster A listed the members of the conic family, explained their family name by mention of their historical origin in the study by the Greeks of the right-angled cone, and showed by diagrams of elliptic and parabolic orbits one of their earliest applications at the hands of Kepler and Newton.

Poster B showed two exact drafting-room constructions for ellipses, one using concentric circles, the other being a projective or so-called "parallelogram" construction.[4]

The student. The student explained the nature of the conics as shown by the string models and amplified briefly the outline of their history and use. He then demonstrated the pin and string construction of the ellipse and the operation of the elliptic trammel, while explaining the frequency with which this curve is met in the drafting room.

SECTION 2. THE ELLIPSE AND PARABOLA (FIG. 2)

On the table. An elliptical "billiard board" made by a student in the school woodshop demonstrated the property of the ellipse of reflecting anything starting from one focus back to the other focus, thus furnishing the principle underlying the construction of whispering galleries, such as that in the Capitol at Washington, D. C. Two focusing flashlights: one disassembled to show the reflector and the movement of the bulb; the other used by the student to demonstrate the result of moving the bulb into or out of the focus of the parabola.

Pictorial display. Poster A contained a diagram of an ellipse showing several tangents and the focal radii to their points of contact, also a picture from *Life*[5] magazine showing an experiment with an elliptical light reflector at the General Electric Nela Park Laboratories.

Poster B showed a projective or parallelogram construction for the parabola. Diagrams showed the use of the focal properties of the parabola in the spotlight and reflecting telescope.

The student. The student demonstrated the elliptical billiard

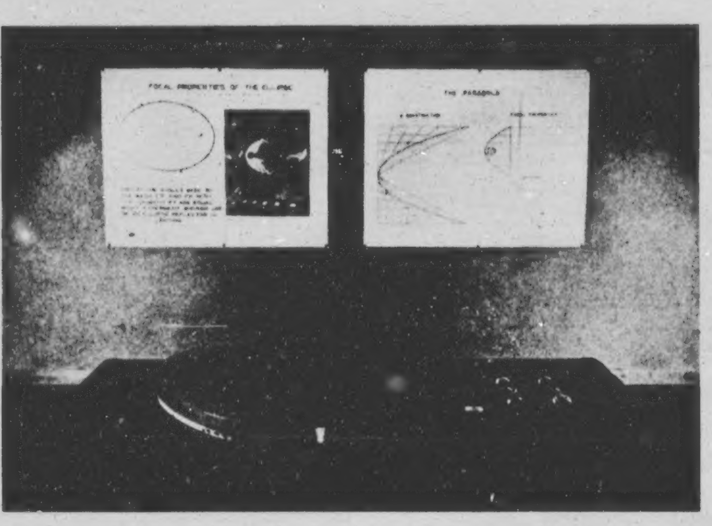

Figure 2. The Ellipse and the Parabola

board and focusing flashlight as a part of an explanation of the focal properties of the two curves and their uses.

SECTION 3. THE PARABOLA, CATENARY, MAXIMA AND MINIMA (FIG. 3)

On the table. Stoppered glass tube partly filled with a colored liquid and mounted so it could be rotated about its vertical axis, thus showing the parabolic shape of the surface of a rotated fluid and demonstrating a device which, when geared to the drivers and properly calibrated, served as a speedometer on some early trains, the speed being determined by the height to which the oil would rise.[6]

It has been suggested to the author that, if mercury were used as the liquid and a light source with nearly parallel rays located above, the focal property of the parabola could be demonstrated and the variation in the location of the focus with a change in the shape of the parabola could be shown by varying the speed of rotation. In either case, a warning is in order to start the apparatus at slow speed and with a small amount of liquid to make sure that the tube is perfectly centered before proceeding.

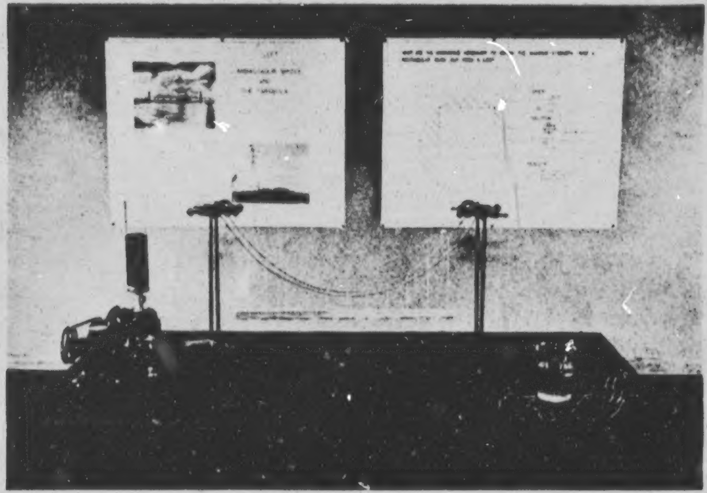

Figure 5. The Parabola, Catenary, Maxima and Minima

Also on the table were two pairs of ringstands of the same height, side by side, from which were suspended two cables of the same length, one loaded horizontally to show a parabolic shape, the other a free-hanging catenoid cable. (To have this set up as accurately as possible, the "cables" should be flexible and the load suspended from the one should be large in relation to the weight of the cable.)

A soap solution and wire forms for demonstrating minimal surfaces, and two tin cans, found in a grocery store, which had the same volume but different shapes and surface areas completed the array of apparatus.

Pictorial display. Poster A showed a picture of the Detroit-Windsor Ambassador Bridge and of a high tension line. Poster B was a diagram for and solution of the problem of determining the dimensions necessary to give the beam of maximum strength which can be cut from a given log.

The student. The student began his discussion by explaining wherein the rotating fluid demonstrated another occurrence of the curve which the spectator had met in the previous section. He then called attention to the parabolic cables of the Detroit-

Windsor Ambassador Bridge[7] but emphasized that free-hanging cables, such as high tension wires, are not parabolic but catenoidal. The catenary, he continued, is a curve with many interesting properties and uses in its own right; for instance, it is the sail curve,[8] as could be shown by fastening a handkerchief to parallel uprights and allowing a fan to blow on it; it likewise provides the basis for computations made by surveyors in correcting for the sag in steel tapes and chains;[9] it finds a place in architecture;[10] and, as demonstrated by the soap solution and wire forms, it provides the curve which, when rotated, traces out the surface of least area which can be used to join parallel circular rings.[11] The problem of "least" and "largest" incidentally, the student explained, occurs in many places in mathematics and its applications, such as in determining the dimensions for the strongest beam that can be cut from a given log or in determining the dimensions for a tin can which will give the minimum area for a fixed volume.

SECTION 4. PLANE AND SPHERICAL TRIGONOMETRY AND GRAPHICAL COMPUTATION (FIG. 4)

On the table. A large model screw thread and a spherical

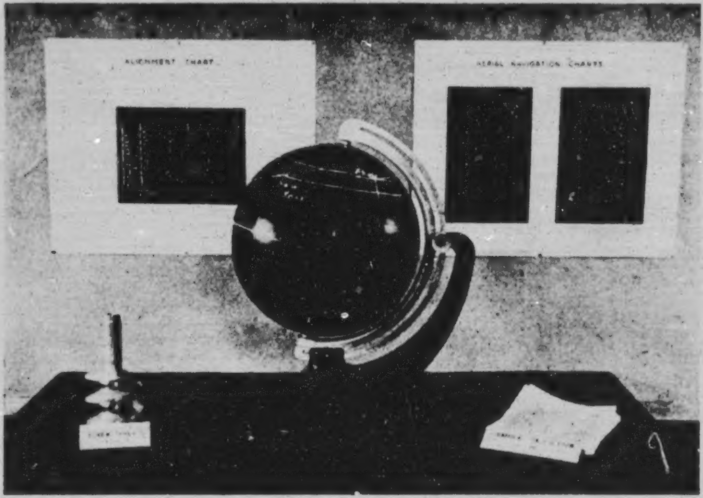

Figure 4. Plane and Spherical Trigonometry and Graphical Computation

blackboard made in the school shops. On the blackboard were laid out the equator, the meridian of Greenwich, and the latitude and longitude lines for New York and Paris; then in colored chalk were drawn the great circle and the loxodrome joining the two cities. At the end of the table, for free distribution, was a pile of mimeographed "Graphical Computation Charts," nomograms on a single sheet for multiplication and division and for determining circumferences and areas of circles.

Pictorial display. Poster A (not shown in Fig. 4) comprised a diagram and solution for determining the helix angle of a U. S. Standard Form Thread; a diagram and solution of the spherical triangle for the distance and bearing of Paris from New York. Poster B showed photostats of alignment charts for tension in and horsepower of belts.[12] Poster C showed photostats of aerial navigation computation charts.[13]

The student. The student first explained the solution of the screw thread problem by plane trigonometry, and then discussed the problem of geodesics,[14] the contrast between great circle and rhumb line sailing, and the use of spherical trigonometry in navigation. He then emphasized the usefulness of graphical computation in industry and navigation and passed out the mimeographed nomograms.

SECTION 5. MECHANICAL COMPUTATION—SLIDE RULE (FIG. 5)

On the table. A set of Napier's bones[15] made by the student

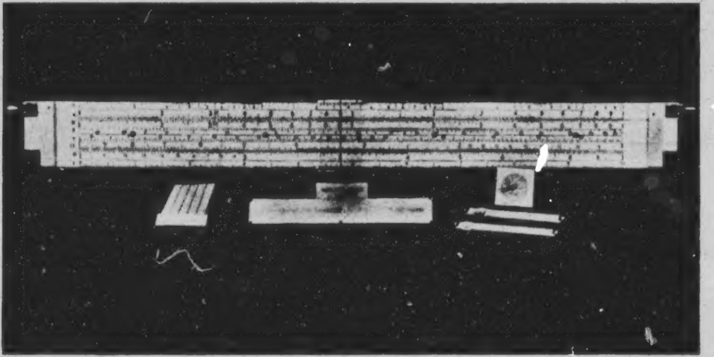

Figure 5. Mechanical Computation—Slide Rule

demonstrator, a Gunter's scale, a circular, a polyphase duplex, and a log-log-trig slide rule.

Pictorial display. A large classroom demonstration slide rule.

The student. The student discussed the historical development of logarithmic scales from Napier and Gunter to the modern slide rule and demonstrated each device.

SECTION 6. MECHANICAL COMPUTATION: PLANIMETER AND CALCULATOR (FIG. 6)

On the table. A polar planimeter set upon a diagram constructed from data of an experiment on hysteresis; an electric calculator.

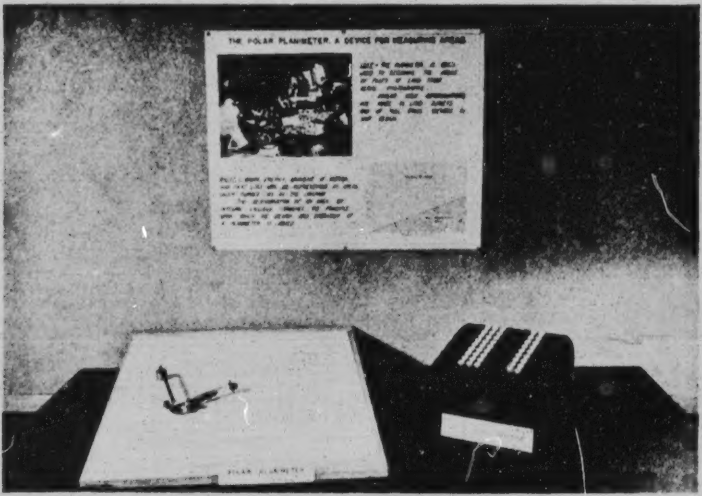

Figure 6. Mechanical Computation: Planimeter and Calculator

Pictorial display. A picture from *Life* magazine showed a planimeter in use in determining areas of wood lots from aerial photographs; graph and discussion of representation of work by an area.

The student. The student explained and demonstrated the two machines and their uses.

SECTION 7. FAMOUS PROBLEMS AND RECREATIONS

On the table. Angle trisector, made in the school machine shop, embodying the method of Archimedes; a rack of books labeled "Mathematics Goes Best Seller," including the well-known biographies and popularizations of the last few years by Bell, Hogben, Kasner and Newman, several works on recreations and puzzles by Ball, Heath, Steinhaus, and a little set of cardboard jigsaws copyrighted under the title "Geometricks," then popular in the stores; and, finally, a pile of mimeographed recreations, puzzles, and games for free distribution.

Pictorial display. The display comprised Poster A, "Solved Problems"—the three famous Greek problems; Poster B, "Unsolved Problems"—Fermat's last theorem and the four-color problem.

The student. The student explained the problems listed on the placards (diagrams were shown for trisection and the four-color problem), demonstrated the angle trisector, and passed out mimeographed material.

The appearance of the placards was improved by making them of uniform size (22″ x 28″) with uniform borders, and by using India ink for all lettering and diagrams (red for construction lines and special diagrams).

The greatest difficulty with the student demonstrators was that when they had worked out their display, made their placards, and studied the historical and theoretical background, they were not only prepared but very willing to talk too long. This danger was overcome by having them stick to a direct and concise explanation of the materials on display, adding historical and other details only as needed to answer questions.

Unquestionably the students who worked hard to put such exhibits into operation had the greatest increase in mathematical knowledge and interest and also the most fun. However, the spectators and fellow students profited through stimulated interest in and added appreciation of the mathematical side of the world about them.

Exhibits of this type often gain in coherence and real teaching values if spectators can be routed through in definite order so that the different sections can be planned as parts of a whole development of some general topic. If this is done, each student demonstrator should take care to point out the relationship of his section to those preceding and following as well as to the general theme of the exhibit.

SUGGESTED TOPICS FOR EXHIBITS

Several unifying topics which might be used as suggestions for either entire exhibits or sections of larger exhibits are briefly described below.

The conic sections. To materials such as those discussed above might be added the following: a model of rolling ellipses or elliptic gears;[3] maps showing the occurrence of conic sections in map projections of the sphere, particularly the gnomonic or great circle sailing charts;[16] a spring gun, a water can with holes punched in the side, or a hose arrangement to show the parabola as the path of a projectile. Further uses of the focal property of the parabola in construction of auditoriums, directional microphones, heat reflectors, etc., could be diagrammed and explained.[17] The derivation of all the conics as sections of a cone of light may be set up with apparatus as simple as a flashlight and a piece of cardboard or be made as complex as one wishes.[18] Conic compasses could be built and demonstrated.[19]

Computation—graphic and mechanical methods, instruments. The material on nomograms, intersection charts, Napier's bones, Gunter's scales, slide rules, calculators pictured and discussed in connection with Sections 4, 5, and 6 could be amplified with demonstrations of the abacus, navigational computers, some of which can be made inexpensively,[20] special slide rules;[21] of aerial navigation problems solved by vector diagrams which could also be made more concrete to spectators by setting up triangles of forces with spring balances from the physics laboratory; of arithmetic short-cuts and discussions of calculating prodigies;[22] and machines in advanced mathematics (harmonic analyzers, integrators, tide calculators, etc.). The use of the carpenter's square, micrometers, ellipsographs, and other measuring, drafting, and calculating tools might be appropriately demonstrated in such an exhibit.

Mechanics, mechanical motions, and machines. A simple mechanical apparatus can be built to draw a spiral of Archimedes[3] and to show its use in uniform motion cams[3] and centrifugal pumps.[3] The limacon of Pascal [3] gives the shape of a simple harmonic motion cam which should also be related to the trigonometric functions and wave motion. Rolling ellipses or elliptic gears have already been mentioned; rolling cones provide a method of obtaining a variety of speeds from uniformly rotating shafts. The involute of a circle[3] may be traced by the unwinding of a string from a circle, and its importance and common occurrence in the design of gear teeth pointed out. The locus of a point on a circle rolling on a straight line is a cycloid,[3] a curve easily demonstrated mechanically or constructed by methods of elementary geometry, which has an interesting history, a modern use in gearing, and a place in theoretical mechanics as a tautochrone[3] and a brachistochrone.[3] There is a considerable amount of elementary trigonometry in screw threads and gears, models of which may be made of wood or metal. Parabolic and catenoidal cables have been mentioned previously; the shape of a beam or of its bending moment diagram bent under some loadings is represented by a cubic or a parabolic curve.

The straight line. The question of how to draw a straight line is related to the question of what a straight line is, and may be used to introduce a discussion of logic and the philosophy of mathematics, as well as to introduce demonstrations of mechanical constructions using the Peaucellier inversor[23] and the hypocycloid traced by a point on a circle rolling on the inside of a second circle of twice the diameter. A variety of linkages giving both approximate and exact parallel and straight-line motions have arisen from the needs of machine design, that of James Watt being of historical interest.[24] Straight-line geometry and projective constructions for curves using only straight lines to locate points or draw tangents are found to be fun for students and of interest to spectators.[25] Related to this topic also is the general problem of shortest distances or geodesics mentioned earlier and also that of the significance of a straight line on a map. This latter is of particular importance today. An explanation with a globe, several maps, and sets of computations of the straight line as representing a rhumb line or constant bearing course on a Mercator map and as representing a great circle on a gnomonic projection will attract interest. This leads naturally to the topic of map projection.

Map projections. This topic would make an entire exhibit, with a discussion of the many different types of maps, their history, their adaptation to particular purposes, and popular misconceptions traceable to them, with globes, drawings, maps, aerial photographs and surveys, the instruments of geodetic surveying, and the nature of the dependence of navigation on maps.[26]

Mathematics and the war.[27] This topic suggests apparatus such as a spring gun and diagrams for demonstrating trajectories and problems in ballistics, range finding and gunnery,[28] aerial photography, navigation, and aeronautics. Probability and statistics play a part in war in a number of places, such as in ballistics and cryptanalysis,[29] and might warrant a separate exhibit.

Probability and statistics. This exhibit can be made interesting through apparatus to show a mechanical construction of the probability curve by chance distribution of balls into cells; through the computation of π by experimentation,[30] discussion of applications in gambling, insurance, popular polls, football forecasts, theory of errors, ballistics, cryptanalysis.

Wave motion. This is the common characteristic of so much in modern physics (and war)—sound, light, radio, heat, fluttering wings, and vibrating propellers—that a demonstration of the wide variety of applications of trigonometric analysis could hardly fail to interest everyone.[31] This can be done by the use of an oscillograph or pencils fastened to vibrating rods or tuning forks, plus superimposed graphs of corresponding fundamental sine waves and harmonics. Simple harmonic motion apparatus discussed in another paper (see page 220) could be adapted to drawing sine waves. An interesting series of relationships is demonstrated by wrapping a paper around a candle, cutting the cylinder at an angle to show an elliptic section, and then unwrapping the paper to show a sine wave.[32]

Every reader will undoubtedly be able to add to those listed briefly above many suggestions for the dramatization of mathematics. A skimming of the recent popularizations of mathematics by Hogben, Kasner and Newman, and Bakst as cited in the notes will suggest other opportunities for dramatization. All that remains is to take action. Try it! It's fun!

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   Drake, R. and Johnson, D., "Vitalizing Geometry with Visual Aids." *The Mathematics Teacher,* Vol. 33, p. 56, February 1940. This article does not deal with exhibits explicitly but lists instruments, models, posters, and activities which would also be excellent for exhibits and includes a good bibliography.
   Gregory, M. Cottell, "A Mathematics Exhibit." *The Mathematics Teacher,* Vol. 23, p. 382, October 1930.
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   Krathwohl, Wm. C., "Using and Preparing a Mathematics Exhibit." *School Science and Mathematics,* Vol. 39, p. 702, November 1939.
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   Sell, Wm., "A Home-Made Mathematics Exhibit." *American Mathematical Monthly,* Vol. 40, p. 555, November 1933.
   Wilson, R., "A Unique Mathematics Exhibit." *The Mathematics Teacher,* Vol. 30, p. 128, March 1937.
2. National Council of Teachers of Mathematics, *The Fifteenth Yearbook, The Place of Mathematics in Secondary Education,* 1940. Appendix I, "Analysis of Mathematical Needs," could also be used as a source of suggestions for themes for exhibits and for practical applications which could be demonstrated.
3. These items of apparatus are more fully pictured and discussed in the article "Mathematical Apparatus," pp. 212 to 225.
4. Any engineering drawing textbook, such as French, T. E., *A Manual of Engineering Drawing* (McGraw-Hill, New York, 1941), and many handbooks will give both approximate and exact constructions for the conic sections and other useful curves. These are within the range of high school and junior high school geometry students and combine practice in the use of instruments with the fun of seeing an old, important, and useful curve grow point by point (or tangent by tangent) under one's hands.

   For constructions of parabolas, see also Wolff, Georg, "The Mathematical Collection," *The Eighth Yearbook of the National Council of Teachers of Mathematics,* 1933, pp. 220, 221.

5. "Light Control Demonstrator Shows How Lenses and Reflectors Work," *Life,* Vol. 8, p. 72, March 11, 1940.
6. Osgood, W. F., *Mechanics* (Macmillan Co., 1937), p. 105, derives the differential equation for this curve and tells of the early use on locomotives. See also Wolff, Georg, *op. cit.*, p. 223.
7. A particularly interesting, popularized discussion by Harold J. Fitzgerald of some of the mathematical problems connected with the Golden Gate Bridge is to be found under the title "All Figured Out," in The Lion's Mouth section of *Harper's Magazine,* October 1936, p. 551.

   For a technical discussion of loaded and free-hanging cables, see Seeley, F. B. and Ensign, N. E., *Analytical Mechanics for Engineers* (Wiley, 1933), pp. 93-96, or any other textbook in applied mechanics.
8. *Encyclopædia Britannica,* 14th Edition, "Curves, Special." Many interesting items on a large number of curves are scattered throughout this article.
9. Huntington, E. V., *Handbook of Mathematics for Engineers,* p. 150. McGraw-Hill Book Co., 1934.
10. Wolff, Georg, "Mathematics as Related to Other Great Fields of Knowledge." *The Eleventh Yearbook of the National Council of Teachers of Mathematics,* 1936, p. 207.
11. For the catenary as a minimal surface, see Bliss, G. A., *Calculus of Variations* (Open Court, 1925), pp. 85 ff.; also Courant, R., "Soap Film Experiments with Minimal Surfaces," *American Mathematical Monthly,* Vol. 47, p. 167, March 1940; Steinhaus, H., *Mathematical Snapshots* (Stechert), p. 107.

    Other references on minimal surfaces are: Douglas, J., "The Problem of Plateau," *Scripta Mathematica,* Vol. 5, p. 159, July 1938; "Soap Films Automatically Solve Problems in Higher Mathematics," *Life,* Vol. 12, p. 118, March 16, 1942.
12. These photostats were from Lipka, J., *Graphical and Mechanical Computation* (Wiley, 1918), an excellent elementary reference on nomography. See also article on "Nomographs in High School Mathematics" in this *Yearbook.* A very nice, though complicated, nomographic chart for elementary computation as well as the quadratic equation is pictured in an article by L. R. Ford, "An Alignment Chart for the Quadratic Equation," *American Mathematical Monthly,* Vol. 46, p. 508, October 1939.
13. These charts were from Lyon, Thoburn C., "Practical Air Navigation," *Civil Aeronautics Bulletin.* No. 24, 1940, pp. 147, 152.
14. Kasner, E. and Newman, J., *Mathematics and the Imagination* (Simon and Schuster, 1940), p. 146, gives a good popularized discussion of the general problem of geodesics and adds (p. 181) a bit of fun interest with its spider and fly problems, as does also Ball, W. W. R., *Mathematical Recreations and Essays* (Macmillan, 1940), p. 118.
15. A translation of Napier's own description of his bones or rods is given in Smith, D. E., *A Source Book in Mathematics* (McGraw-Hill, 1929),

p. 182; Bakst, A., *Mathematics, Its Magic and Mastery* (Van Nostrand, 1941), p. 117. In the same chapter are also discussions of the abacus and other computing devices as well as arithmetic short-cuts.

16. Bradley, A. D., *Mathematics of Map Projections and Navigation,* pp. 24 ff. Yoder Instruments, 1938.
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18. Hurlburt, Everett H., "A Simple Optical Device for Demonstrating the Conic Sections," *School Science and Mathematics,* Vol. 41, p. 828, December 1941. Tells of an interesting spectator-operated device, wherein the guest picks his curve, pushes the button, and the device does the rest.
19. Lof, J. L. C., "The Conic Compass," *School Science and Mathematics,* Vol. 38, p. 842, November 1938.
20. A cardboard and celluloid "computer" ready to be cut out and assembled comes with Lyons, T. C., *Practical Air Navigation.* This computer consists of a circular slide rule with special markings and an arrangement of circular grids giving graphical solutions for vector triangle problems involving air speed, wind velocity, course, and ground speed.
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22. Ball, W. W. R., *Mathematical Recreations and Essays,* Chap. XIII, pp. 350 ff. Macmillan Co., 1940.
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24. Keown, R. M. and Faires, V. M., *Mechanism,* pp. 11 ff. McGraw-Hill Book Co., 1939.
25. See texts in projective geometry, French, *op. cit.;* Yates, *op. cit.;* "Harvard University Sophomore Makes Line Drawings," *Life,* March 18, 1940, p. 43.
26. Bradley, A. D., *op. cit.* Also, by the same author, "Gnomonic Projection of the Sphere," *American Mathematical Monthly,* Vol. 47, p. 694, December 1940. Deetz, C. H. and Adams, O. S., "Elements of Map Projections," *Department of Commerce, Coast and Geodetic Survey Special Bulletin,* No. 68, 1938. "Maps," *Life,* 13:5, pp. 57 ff. August 3, 1942. Steinhaus, H., *op. cit.*, pp. 94 ff. Bauer, H. A., *Globes, Maps, and Skyways.* Macmillan Co., 1942.
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University Center for graduate study in applied mathematics in defense is discussed on pages 238 and 330 respectively of these two issues). See also: United States Military Academy, *Some Military Applications of Elementary Mathematics*, 1942. An interesting mimeographed collection with diagrams and some aerial photographs.

28. "Field Artillery—How to Fire a 75 mm. Gun," *Life*, Vol. 10, p. 64, February 10, 1941. Thomas, J. M., *Elementary Mathematics in Artillery Fire*. McGraw-Hill Book Co., 1941. Levy, S. H., *Introductory Artillery Mathematics and Antiaircraft Mathematics*. University of California Press, 1943.
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31. Karpinski, Benedict, Calhoun, *op. cit.*, Chap. XXVI, p. 407.
32. Karpinski, Benedict, Calhoun, *op. cit.*, p. 419. Steinhaus, H., *op. cit.*, p. 59.

EAST HIGH SCHOOL VITALIZES MATHEMATICS

H. W. Charlesworth

For a number of years East High School of Denver has been holding a mathematics exhibit. These annual events have become quite spectacular, each year growing in importance as well as in size. The Fourth Annual Mathematics Exhibit of April 8 to 11, 1942, filled the entire floor space of the boys' gymnasium, which was attractively decorated in red, white, and blue. Since these exhibits were open during school hours and at least two evenings, they were seen by the entire student body and visited by several hundred patrons and out-of-town teachers and their pupils. A visitor to one of these exhibits could easily imagine himself to be at a world's fair.

Sponsorship of the Exhibits

The exhibits were started and are still sponsored by the Euclidian Club, the mathematics club of the school. Because this club wished to give mathematics the prominence it deserves in a modern high school, the exhibit was chosen as one means of showing the value and importance of mathematics in modern life. It has proved to be more than was hoped for. Now it is listed in the student handbook as one of the traditions of the school.

These exhibits require a great deal of planning and work on the part of teachers and pupils. They are a cooperative school endeavor. Any pupil, group of pupils, class or club may enter a project, serve on committees, or help in some way. The real success of the exhibit depends upon student participation. More than three hundred pupils shared in some way in making the exhibit of 1942.

The exhibits are financed by the sale of souvenir pencils and magic slates and by money donations from the various school organizations and clubs. The exhibit in 1942 cost approximately $150.

General View of Exhibit Room. Theme: Mathematics, A Universal Language

Mathematics and Light—A Part of Exhibit on Mathematics and Physical Science. Booth Prepared by Four Boys

THE DISPLAY

The exhibit is the result of pupil study and research. The one stipulation is that the exhibit *must* show the applications of mathematics. It must *not* be a display of classroom work or a collection of papers or notebooks. Pupils work up a project for the exhibit, alone or in a group. Although actual work is done outside class time, it is expected that many projects will originate in the classroom work.

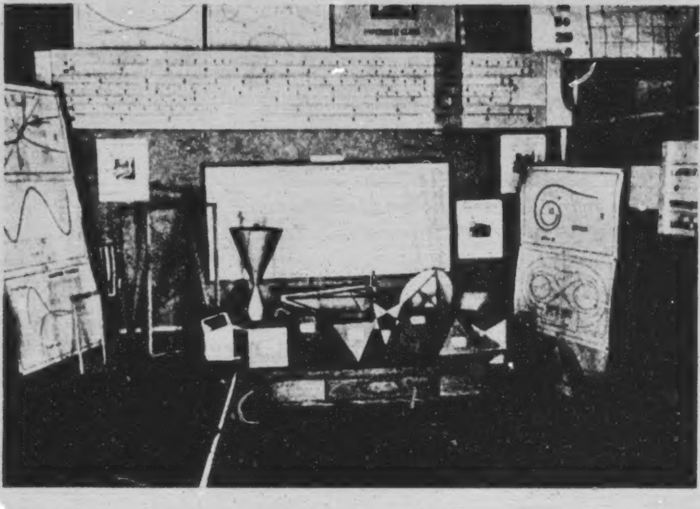
The project may be worked out in any field, provided that the purpose is to show the use of mathematics. The project may be original or not. It may be the result of a hobby, provided, of course, that it gives prominence to the use of mathematics. It is expected that pupils work with the advice and help of teachers and others.

The results of the project must be colorful and displayed attractively, and must be accompanied by appropriate posters and explanatory materials. Much freedom is allowed in this display, and pupils are encouraged to decorate the exhibit space allotted to them. A group project, one involving five or more pupils, a class or a club, is given an exhibit space of at least eight feet by ten feet.

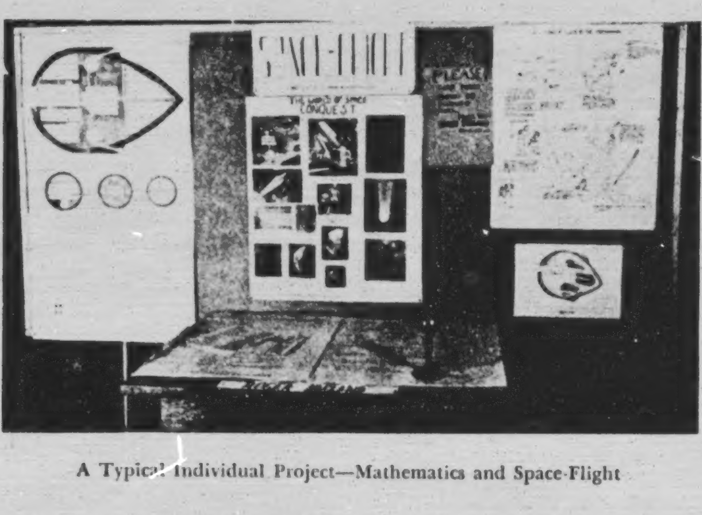
The cooperation of local business and industrial concerns is encouraged, provided no advertising as such is done. Pupils who get help from community organizations are expected to study and work out the problem for themselves. We want it to be an educative process. The cooperation of out-of-school groups has been very satisfactory and has proved to be a great help in relating school and community in mutually beneficial ways.

THEME AND DECORATIONS

Every annual exhibit has a theme. Two recent ones have been "Mathematics—A Universal Language" and "Mathematics in the World of Tomorrow." We symbolize the theme by a large central exhibit piece, usually a composite geometric solid occupying floor space of about twelve feet by twelve feet and standing from fourteen to twenty feet high. In 1942 we constructed a stellated icosahedron about thirteen feet in width; this was suspended from the ceiling and made to revolve, apparently revolving on the apex



Booth Prepared by Five Boys with Help of Colorado School of Mines



SPACE-FLIGHT
THE LORDS OF SPACE CONQUEST
PLEASE

A Typical Individual Project—Mathematics and Space-Flight of a square pyramid. This piece was done in red, white, and blue to conform to the general scheme of decoration, as were also the booths and other exhibit units. Significant wording was printed on the four lateral faces of the pyramid.

VALUES DERIVED FROM THE EXHIBITS

The values originally set have been realized, many of them far beyond expectations. We are learning how to bring about and develop other values, some of which are not obvious and are slow in development. These annual exhibits bring our mathematics department into close contact with other departments of the school, which cooperate in a very fine way. The exhibits have also helped to improve community and school relationships.

Despite certain curricula changes that have tended to discourage the election of mathematics courses, the subject has more than held its own in our school. We believe that these exhibits have helped greatly. The values to the participants cannot be measured, but we are confident that they far exceed the actual cost. It is true that the exhibits mean a great deal of work, but we heartily recommend them to any school desirous of keeping mathematics in its proper place of prominence in a modern high school.

RECORDING THE EXHIBIT

Each year we print a mimeographed booklet which tells how we plan and conduct our exhibits. It gives brief descriptions of several typical projects and lists many others. (Supply exhausted.)

We made movies of the 1940 and 1942 exhibits. The latter was a student project from beginning to end, constituting one of the major projects of the exhibit. The group of students who made the movie were allowed twenty-five dollars for the entire job, and they did it for slightly less. Nine hundred feet of film were exposed, about five hundred feet of which were retained. It is a good picture, a 16-mm silent film accompanied by a typed narrative. This film, *Mathematics in the World of Tomorrow*, will be lent for the cost of transportation both ways. The mimeographed booklet, together with the narrated film, gives quite a complete story of our annual exhibits.

EMPLOYING VISUAL AIDS TO TEACH LOCUS IN PLANE GEOMETRY

Edna Hitchcock Young

THE TOPIC "locus" is truly a difficult one in the teaching of plane geometry. No doubt many teachers and students have emerged from such a unit both discouraged and confused.

The usual definition, "A locus is the path of a point which moves so that it always satisfies a given geometric condition," is itself meaningless to the majority of students. Up to this point in the course each figure in geometry has been static; then suddenly a concept of motion is introduced. To add further to his bewilderment the student is expected to be able to prove that every point on a specific locus satisfies the given condition and either that every point which does satisfy the condition is on the locus or that any point not on the locus cannot satisfy the condition. After completing the work on locus, both teacher and student experience a mental jolt when they view the results of a simple test on perhaps only the fundamentals of locus. It is safe to predict that the resulting figures will range from circles and parallel lines to cycloids and rectangles and even some indescribable figures—to say nothing of the accompanying proof, if the student has enough mental fortitude to attempt one.

It is a natural temptation to omit the unit entirely or to skim over it superficially, thereby reducing its demoralizing effects to a minimum. Yet the unit on locus is a bridge between the study of the theories of geometry and their practical application in the fields of mechanics, construction, navigation, and analytic geometry. It is also valuable as a comprehensive review of the relation of many geometric facts to each other.

It was a determination to preserve the study of locus that grew into an experience most satisfying to me. In the spring of 1936, a neophyte teacher, I was greeted by a class of some thirty students who had been weeded from the classes of three other teachers in an attempt to reduce excessive teaching loads. I would not have admitted it then, but I will now—I was panic-stricken and therefore I clung firmly to the text. Things began to run smoothly toward the end of the unit on circles, but I hate to recall the subsequent work on locus—it exhausted me and "finished" that class so far as enjoyment of geometry was concerned. No matter how much I tried, I could not recapture their interest.

By fall I had recovered from my case of "locophobia" and returned to school resolved to oust this dread disease from the classroom. With the very first constructions, such as angle bisectors and perpendicular bisectors, the term "locus" was introduced, this time with emphasis on its derivation from the Latin word meaning "place." "Where are some points which are each equidistant from two fixed points *A* and *B*?" I asked. After marking several such points with compasses, the students discovered that the "place" or the "locus" of all of them was the perpendicular bisector of *AB*. It was quite obvious to them that any point on this perpendicular bisector was equidistant from *A* and *B*, and that every point elsewhere was nearer to either *A* or *B*. This procedure of talking about locus was injected into every construction problem from the beginning of our study, so that with the arrival of the unit on locus the term itself was nothing new.

To launch the unit with enthusiasm and to introduce the concept of motion, we tried to visualize the path of the center of an automobile wheel as it rolled along the street. This was not difficult, but the path of a point on the circumference of the same wheel proved more stimulating. Heated arguments followed; some students thought the locus would be a straight line parallel to the surface of the road; others suggested that it would be a circle; still others imagined a series of loops like an old-fashioned penmanship exercise. Even after they had gone home and observed actual wheels in motion and experimented with their mother's chinaware and pantry equipment, there was still a difference of opinion. Finally one enterprising lad—I wish I could recall his name, for I am greatly indebted to him—conceived the idea of bringing a model into the classroom to terminate the controversy. He made one of a wooden wheel, with a radius of about four inches, which rolled along a straight track. By inserting a pencil near the circle, he actually traced a cycloid on a board mounted behind the revolving wheel. This model, although crude, was convincing.

The discussion of the locus of the center of a circle of given radius which remains tangent to the sides of a triangle or rectangle brought forth differences of opinion again. But the idea of making models had taken hold and soon there were many models, not the elaborate polished variety, but simple cardboard models to be used today and discarded tomorrow. Some students ventured to apply the same method to squares, pentagons, and regular hexagons, as shown in Figure 1, at the left, below. In each case a model displayed why the internal figure is a polygon, while the external figure has rounded vertices.

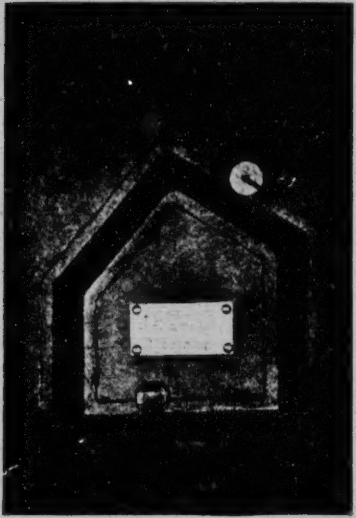


Figure 1

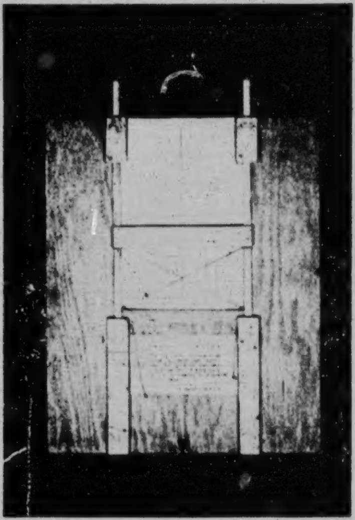


Figure 2

More and more models were brought to class; interest in them seemed to grow. Many students told of the construction difficulties they had encountered and the ways in which they had overcome them. Some were really ingenious. For example, while using the model for the cycloid, the class observed that it was difficult to make the wheel roll, for it had a tendency to slide on the track.

Many suggestions were made, such as stretching a piece of inner tube around the wheel to increase its traction on the track. The result was an improved model, but one still imperfect. Someone suggested the use of a strip of corrugated cardboard on the outer rim of the wheel and a strip of the same cardboard on the track. It was found that stretching the corrugations on the curved surface prevented perfect synchronization with the cogs on the flat surface. Then spikes were inserted on the surface of the wheel and the track was replaced by a soft, easily penetrated material. This time it was observed that the spikes on the wheel had to enter and leave the track at angles instead of along the lines perpendicular to the track. This difficulty may have been overcome, but the problem was solved when a student found that a piece of flexible wire held securely in a position parallel to the track and wrapped tautly around the axle of the wheel prevented its sliding and made it revolve as it progressed along the track.

Enthusiasm for models need not weaken the demonstration phase of the work. Many students, once understanding the meaning of locus, prefer to imagine the path of the point or to discover its form by referring to related theorems; others continue to depend upon a model as an aid. For example, alert students would associate the locus of the point of intersection of the diagonals of a rectangle having one side fixed in length and position, with the perpendicular bisector of the fixed line. However, many students lacking this facility to associate ideas abstractly would realize that the problem becomes the locus of a point equidistant from the extremities of a line, upon examination of a model such as the one shown in Figure 2. Although my experience has been too limited to justify generalization, I have found that frequently the better students prepare the poorer models. Often they have fine ideas but seem to lack the mechanical skill to execute them, while some of the poorer students create excellent projects. Perhaps this is the natural result of the proportionally greater number of poorer students participating in "shop" courses. However, this difference in ability is most helpful in maintaining a balance between the discussion of models and geometric demonstrations.

To clarify this point, let us consider an example. A problem frequently encountered is to find the locus of the mid-point of a given line-segment which moves so that its extremities remain on the sides of a right angle. The alert student immediately associates this problem with the theorem that the mid-point of the hypotenuse is equidistant from the three vertices; hence he knows that the mid-point must always be one-half the length of the given line-segment from the vertex of the right angle, and that it is therefore an arc of a circle with its center at the vertex and its radius equal to one-half the given line-segment. It would be criminal to stifle his clear thinking and imagination by compelling him to make a model. At the same time there are students incapable of this kind of abstract thinking who, with the cover of a pasteboard box and a ruler or short stick, can discover for themselves the form of the locus and can see the concrete application of the theorem. Frequently a model inspires the creation of a new problem, such as the locus of a fixed point other than the mid-point of given line-segment which moves with its extremities on two perpendicular lines.

Thus the two extremes of ability can work together on the same unit, each contributing to the growth of the other through verbal demonstration or physical evidence.

It is somewhat difficult for the teacher to decide whether models should be required, whether proofs should be required, and how to measure the individual's achievement. These problems, like all other problems in teaching, can be answered only in terms of the abilities, interests, temperaments, and aspirations of the individual members of the class. It seems advisable to allow the student considerable freedom in the selection of the particular kind of work he will do; but, bright or dull, he will need guidance in his efforts.

It has been my experience that the following procedure is helpful. First, the term locus is introduced as soon as possible in conjunction with the most simple construction problems so that, when the locus unit itself is approached, the term is already understood. Since the Pythagorean Theorem may be applied to several locus problems, especially those involving circles with chords or tangents; and since the unit itself is valuable as a comprehensive review, it is best to postpone the locus chapter until well near the end of the course. A discussion of the cycloid or the cardioid stimulates interest in the unit and challenges even the better students to build models. If the students fail to take the initiative, a few models from previous years are brought into the classroom in order to motivate greater interest. Students like to manipulate things, and soon there is a discussion about how to improve one or what principles are involved in the other. However, only a few models are carried from one class into another, for I firmly believe that great value is derived from a child's studying a problem, planning a model, and carrying these plans through to a finished product. After we have discussed many problems, each student is requested to select a specific problem upon which he is to become an authority. He may do one of two things: he may describe the locus and prove it quite formally to the class; or he may prepare a model and demonstrate to the class how it operates and what theorems it illustrates. The first procedure is within the range of only the better students, but through them the entire group becomes acquainted with at least the essence of the geometric proof of a locus problem.

The second procedure is open to all students, for the only materials necessary can be found in their own homes or the school's industrial arts shop. An embroidery hoop or a coffee-can cover makes an excellent circle. In Figure 3 a can cover was used to find the locus of the mid-point of all chords of fixed length in a given circle. The chord is a strip of wood with a piece of lead fastened at its mid-point. Figure 4 shows an embroidery hoop being used as the fixed circle in finding the locus of the center of a circle of given radius, which remains tangent to the fixed circle. In this model the student made use of the line of centers in order to show both internally and externally tangent circles.

A simple yet very effective model can be made entirely of cardboard. In order to show the locus of the vertex of a right angle whose sides pass through two fixed points, it is necessary only to mark the two points on cardboard and cut a slit from one to the other. A second piece of cardboard, cut to a right angle, inserted in the opening is movable. By rotating the right angle and pressing it firmly against the ends of the slit, the student sees that the vertex traces a semicircle. Two pins, one each at the two given

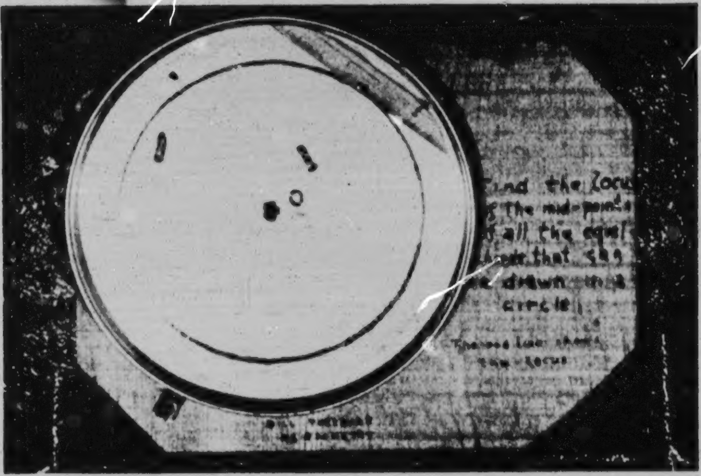

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Figure 3

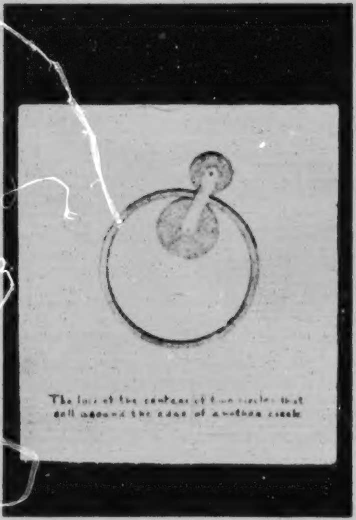

Figure 4

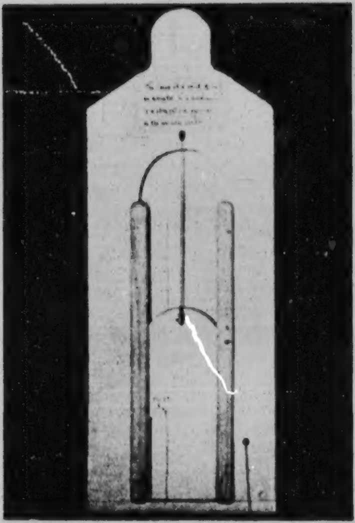

Figure 5

points, may be substituted for the slit, and acute or obtuse angles may be substituted for the right angle. Each of these may be used as the vehicle for the application of the basic theorems about inscribed angles.

Thus wires, strings, tin cans, rubber bands, pencils, nails, pressed wood, and countless other common materials find their way into the classroom in the forms of triangles, circles, parallel lines, elevator shafts, as shown in Figure 5. The models need not be permanently finished products, for only a few are to be preserved. However, frequently very fine models are prepared and these are useful in motivating the work of future classes or in suggesting possible variations to the students of less creative initiative.

The final emergence from this unit is no longer accompanied by a distaste for everything mathematical. True, students may still be unable to write a formal proof, but they have experienced the actual application of some of the basic principles of geometry and have gained an insight into the relationship of one part of a geometric figure to another.

Nothing lost by the better student; something gained by the poorer student! Is it not possible that models could be used elsewhere in mathematics to increase the efficiency and pleasure of both the teacher and the learner?

LINKAGES

Robert C. Yates

THE MOST PROMINENT motion is circular. The conversion of the easily attained circular motion into motion along a straight-line is of prime importance to the engineer and the mechanic. This was especially true seventy-five years ago when modern machinery was in its formative stage. Steam had only recently been applied to both land and water vehicles, but poorly riveted boilers and clumsy levers improperly lubricated played havoc with life and limb.

The generation of line motion was no doubt of concern to mathematicians from the time of Archimedes and, because no solution was apparent, many confused this problem with that of squaring the circle. A solution was first given by Sarrus in 1853 and another by Peaucellier in 1864, both of which lay unnoticed until Lipkin, a student of Tschebyschef, independently recreated Peaucellier's mechanism.

Fanned by Sylvester's enthusiasm [14],* interest in general linkwork immediately flamed high to attract the attention of men like Cayley, Kempe, Hart, Darboux, Clifford, Koenigs, Sir William Thompson, Darwin, Mannheim, and a host of other minds. The epidemic was so fierce and so universal that the subject was drained almost completely dry in the short span of four or five years. The drop in interest followed Sylvester's departure for America and Kempe's proof of the remarkable theorem that any algebraic curve, no matter how complex, can be described by a linkage [8, 15].

We shall confine our attention for the most part to linkages that produce line motion. Although the drawings given here seem to indicate otherwise, there is no necessity that bars or links be straight. Indeed, such a requirement would beg the question. The line segment joining two joints is the effective distance—the only requirement is that all bars be plane inextensible members.

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* Numbers in brackets refer to references on page 129.

In making models of the various linkages, one should obtain colored cardboard (poster board) about 12-ply, an eyelet punch, and boxes of No. 2 and No. 3 eyelets. Use the No. 2 eyelet to join two links; the No. 3 to join three or four links. With a phototrimmer cut the cardboard into strips about one-half inch wide, and mount the model on a cardboard background. To insure greater accuracy, two bars of the same length should be punched simultaneously [15].

THE PEAUCELLIER CELL

The Peaucellier cells [5, 11, 15] are displayed in Figure 1, where $OA = OB = AR = BR = a$; $AP = PB = BQ = QA = b$. It is obvious that the points O, P, Q, as well as points O, P, R, remain collinear as the cells are deformed. If X be the mid-point of the variable distances PQ and OR, we have in either figure:

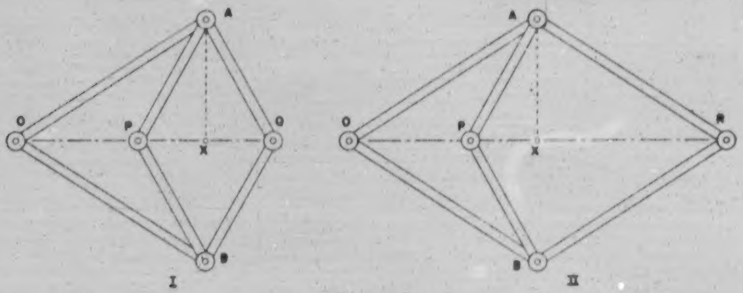

Figure 1

$(OA)^2 - (AX)^2 = (OX)^2$ and $(AP)^2 - (AX)^2 = (PX)^2$.

Subtracting:

$$(OX)^2 - (PX)^2 = (OA)^2 - (AP)^2 = a^2 - b^2,$$

or

$$(OX - PX)(OX + PX) = a^2 - b^2.$$

Thus, in the left figure, since $PX = XQ$ and $OX + PX = OX + XQ = OQ$,

$$(OP)(OQ) = a^2 - b^2.$$

In the right figure, $OX + PX = RX + PX = PR$,

$$(OP)(PR) = a^2 - b^2.$$

These relations constitute the fundamental property of the Peaucellier cells, a property that characterizes them immediately as mechanical *inversors* [15].

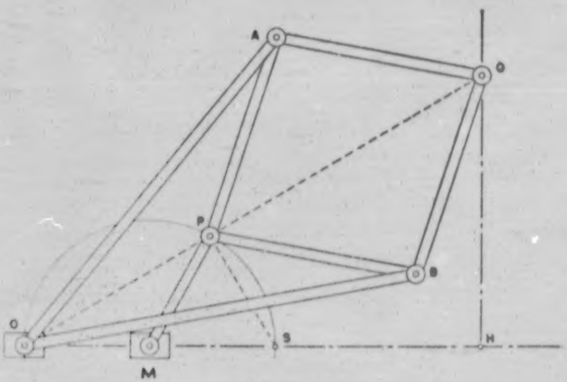

Figure 2

For line motion, a seventh bar of length c is attached with one end at P and the other end, M, to the base plane. The point O is attached to the plane so that $OM = c$. Then Q describes a straight line perpendicular to OM as shown in Figure 2. To prove this, let the linkage be placed in an arbitrary position as indicated. Draw a line through Q perpendicular to the line OM of fixed points. It is evident, since the right triangles OSP and OHQ are similar, that

$$OP/OH = OS/OQ \qquad \text{or} \qquad (OH)(OS) = (OP)(OQ).$$

But $(OP)(OQ)$ is constant and so therefore is $(OH)(OS)$. Thus, since S is a fixed point and this product is constant, H is accordingly fixed and the point Q lies always on the perpendicular to OM at H.

The other arrangement of the Peaucellier cell will produce line motion if P is attached to the plane and O is made to traverse a circle passing through P. The demonstration is similar to the foregoing.

A line motion linkage announced by A. B. Kempe [6, 7, 8, 15] is built from portions of the Peaucellier cell. The "kites" *APRT* and $RO'ST$ of Figure 3 are proportional. T[illegible]t is, $AP = AT$, $PR = RO' = RT$, $O'S = ST$, and

$$AP/PR = PR/ST.$$

Since the two kites are in union as shown, they have a common angle at T and thus are similar throughout all deformations.

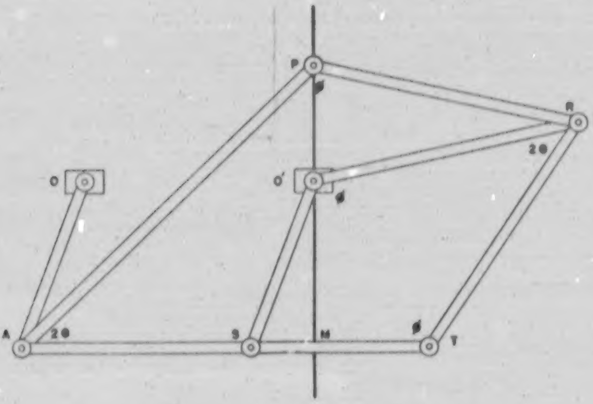

Figure 3

Let $\angle TAP = 2\theta = \angle TRO'$ and $\angle APR = \varphi = \angle ATR = \angle RO'S$. Then

$$\angle PRO' = 2\pi - 2\varphi - 4\theta.$$

Since triangle $PO'R$ is isosceles, $\angle RPO' = \varphi + 2\theta - \frac{\pi}{2}$,

and therefore $$\angle APO' = \frac{\pi}{2} - 2\theta.$$

Thus, since $\angle PAT = 2\theta$, PAM is a right triangle with the line joining P and O' always perpendicular to the bar AT. Accordingly, if we fix O' and move AT parallel to itself, then P will describe a line perpendicular to AT. To do this, attach a bar OA equal in length and parallel to $O'S$ and fix the point O.

Remove the bar OA, free the point O' from the plane, and fix P. Then attach to T one end of a bar which is equal in length to AP. Fix its other end to the plane at Q so that $PQ = AT$. This arrangement permits O' to move on a line perpendicular to PQ and the motion is somewhat freer than the foregoing.

THE HART CELL

A linkage of special importance is the four-bar *crossed parallelogram* of Hart [1, 8, 15]. The bars are equal in pairs, i.e., (Fig. 4), $AD = BC$, $AB = CD$. We select four points O, P, Q, R on the bars in a line parallel to BD and AC. These will remain collinear as the cell is deformed. Draw the circle through A, P, and Q. Since its center is on the perpendicular bisector of PQ, the line of symmetry of A and C, then C also lies on the circle. Let the circle cut the bar AD in the point T. This point is a fixed point of the bar. For, by the secant property of the circle,

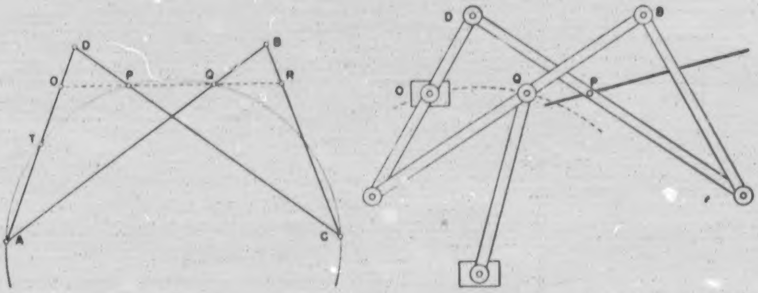

Figure 4

$$(DT)(DA) = (DP)(DC),$$

in which the right member is a constant since D, P, and C are fixed points of the bar. In the left member, DA is constant, and thus T must be a fixed point of AD. That is, throughout all deformations, A, T, P, Q, C, points fixed on the several bars, are always concyclic.

Now, since O is a fixed point of the bar AD, we have also by the secant property:

$$(OP)(OQ) = (OT)(OA) = \text{constant}.$$

Thus, since the product of the variable distances OP and OQ is constant, this remarkable four-bar mechanism has the same fundamental property as that of the Peaucellier cell. For line motion, following the principles of Figure 2, we may fix O to the base plane and cause Q to move on a circle through O. Thus P describes a straight line.

The Hart cell has further interest. If we unionize two such cells by utilizing a short bar of one as a long bar of the other, as shown in Figure 5, we may produce linear motion upon an entirely different principle [6, 7, 8, 15]. In the left-hand figure, $ABCD$ and $ADFE$ are both crossed parallelograms. The angles of the first at A and C (i.e., θ) are always equal; those of the second at A and F (i.e., φ) are likewise equal. Generally, however, $\theta \neq \varphi$. But if θ is to equal φ throughout all deformations, the two crossed parallelograms must be similar and their corresponding sides proportional. That is, if $\theta = \varphi$, then

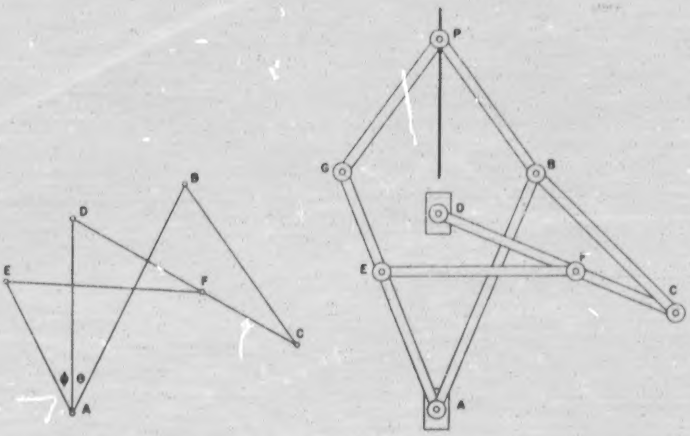

Figure 5

$$AE/AD = AD/AB,$$

or

$$(AD)^2 = (AE) \cdot (AB)$$

Accordingly, if the bar AD be fixed to a base plane, *the rotation of AB about A produces an equal and opposite rotation of AE about A*. Thus, if AE be extended to G such that $AG = AB$ and the two equal bars GP and BP be added, the point P will be forced to move along the straight line passing through the two fixed points A and D.

Special note should be made of the connection here with the Kempe linkage for trisecting the general angle. It is obvious in the left figure that two more bars may be attached to the linkage to produce *three* equal angles at A—the three unionized crossed parallelograms having their corresponding sides proportional. (See article on *Trisection* elsewhere in this volume.)

OTHER LINE MOTION LINKAGES

We present here two linkages whose underlying principles differ from those of the foregoing. The first mechanism, outstanding for its ingenuity and simplicity, was devised by Hart [4, 5, 15]. Two sets of equal bars, $AC = BD$, $PC = PD$, are joined as shown.

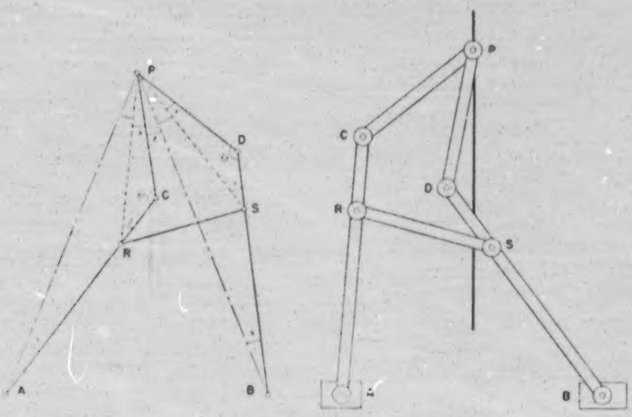

Figure 6

The points A and B are fixed to a base plane at a distance h apart. If P be moved so that the angles at C and D are always equal, then triangles APC and BPD will be congruent and AP always equal to BP. Thus P will lie always on the perpendicular bisector of the segment AB.

The requirement that angles at C and D be always equal would seem difficult indeed to arrange mechanically. Surprisingly enough such is not the case. For, let $AC = BD = a$, $PC = PD = b$. Then select two points R and S on AC and BD, respectively, so that

$$RC = SD = b^2/a.$$

Then $$RC/PC = (b^2/a)/b = b/a = PC/AC.$$

Thus $\angle PAC = \angle RPC = \angle SPD = \angle PBD = x$. Furthermore, $\angle APC = \angle PRC = \angle BPD = y$. Now since $PR = PS$; $PA = PB$; and $y + z = \angle APB$, then triangles APB and RPS are similar. Accordingly,

$$PR/PA = RS/AB = RC/PC = b/a.$$

If we take the constant distance AB equal to a, then

$$RS = b.$$

That is, if P describes the bisecting line of AB, then the distance between the moving points R and S is constant. Conversely, the angles at C and D will remain equal and the point P will describe a line if R and S are joined by a bar of the proper length.

Before attaching the linkage to a base plane, lay it open so that P is at the uppermost point. The mechanism forms the letter "A". In this extreme position fix the points A and B along any desired line.

The second linkage for line motion is due to Kempe [8, 15]. As shown in Figure 7, the following lengths are selected: $AB = BC = CD = CP = 4a$; $AD = DC' = C'D' = C'P = 2a$; $AD' = a$.

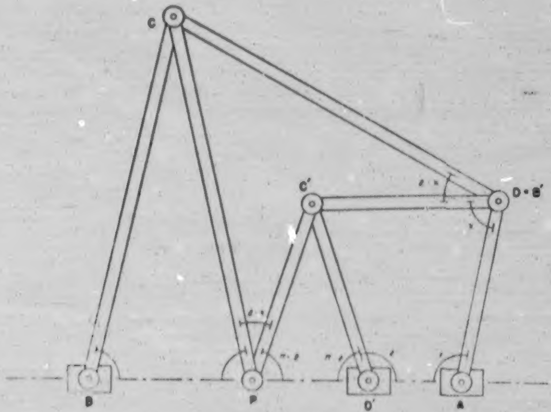

Figure 7

The points A, D', and B are attached in a straight line to the base plane. We shall show that P lies always on this line.[1]

From the selected lengths, quadrilaterals $ABCD$ and $ADC'D'$ are similar, since they contain a common angle y at A. Thus, angles of the first quadrilateral, i.e., x, y, z, are equal to those of the second at corresponding vertices. Moreover,

$$\angle ADC' = x \quad \text{and} \quad \angle C'DC = z - x = \angle C'PC.$$

But $$\angle AD'C' = z \quad \text{and} \quad \angle C'D'P = \pi - z = \angle C'PD'.$$

Accordingly, $$\angle CPD' = (z - x) + (\pi - z) = \pi - x,$$

and therefore the points B, P, and D' are collinear. Consequently, P must move on the straight line $AD'B$.

It should be noted that any point on the bar PC, other than C or P, describes an ellipse.

APPROXIMATE LINE MOTION

The linkages discussed thus far have yielded *exact* solutions of the problem of converting from circular into straight-line motion. Prior to the appearance of the Peaucellier cell, there were, naturally, a number of devices that produced motion along an approximately straight line. The most notable of these was the three-bar linkage of James Watt, father of the modern steam engine [6, 15]. This is displayed in Figure 8 I. P is the mid-point of CD and $AC = BD$. A and B are attached to the plane so that CD is perpendicular to AC and BD when they are parallel. Watt reasoned that the circular motion of C counteracted that of D so that a point P midway on the traversing bar would travel along a "neutral" path approximating a straight line. This device enabled him to reduce the height of his engine house by five feet, a step which must have represented a capital saving. A curious quotation from a letter of 1784 to his son reveals his enthusiasm: ". . . though I am not over-anxious after fame, I am more proud of the parallel motion than of any other invention I have ever made."

[1] The requirement that A, D', and B be collinear would seem to beg the question. However, since it is possible by the compasses alone to establish such points, an objection on this score could not be justified. (See article on *Geometrical Tools*, (pp. 204-211.)

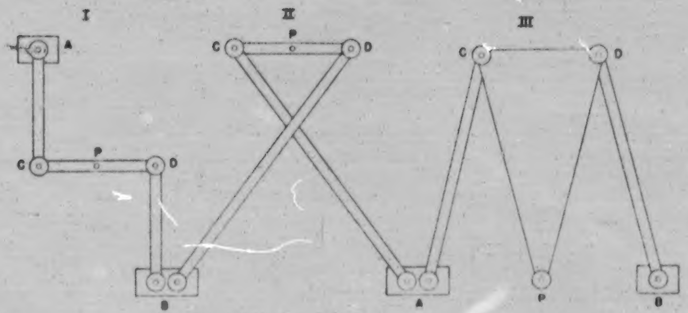

Figure 8

An interesting special case arises when it happens that $AB = 2a$; $CP = PD = a$; $AC = BD = a\sqrt{2}$. Then the locus of P is the familiar lemniscate.

The arrangement of three bars in Figure 8 II is the approximate line motion of Tschebychef given in 1850. Here, $AB = 4a$; $DP = PC = a$; $AC = BD = 5a$.

In 1860, R. Roberts devised the three-bar linkage (Fig. 8 III) which produces a better approximation to line motion than either of the two foregoing mechanisms. Instead of a traversing bar, there is the plate CPD which carries the tracing point P. Lengths are as follows: $AC = CP = PD = DB$ and $AB = 2(CD)$.

We cannot pass on without mentioning the general three-bar linkage shown in Figure 9 I. This mechanism produces a compli-

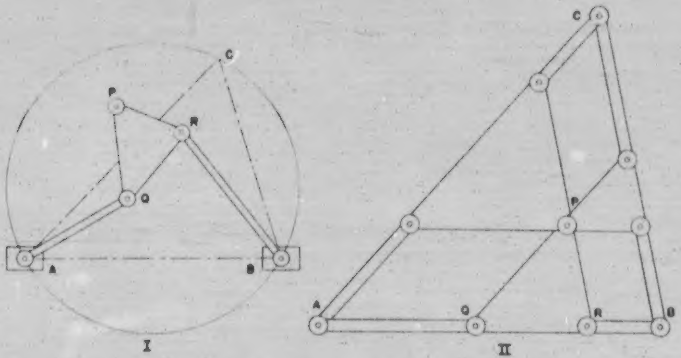

Figure 9

cated curve of the sixth degree [10]. If the triangle *ABC* be drawn similar to the plate *PQR*, the circumcircle of *ABC* will pass through the *double points* of this sextic curve.

Figure 9 II exhibits a most remarkable property of the three-bar linkage. Select a triangle *ABC* and any internal point *P*. Draw lines through *P* parallel to the sides of *ABC*, thus determining a triple three-bar mechanism as shown. (The three-bar part *ABPQR* of I, for example, might be the same as that in II when extended.) Now, no matter how the linkwork be deformed, *triangle ABC remains always similar to itself*. That is, for instance, if *A* and *B* are fixed, *P* describes a three-bar curve, and the free point *C* remains at rest. Or, if *A*, *B*, and *C* are fixed to the plane, *all three of the three-bar mechanisms produce the same curve in mutual harmony and cooperation.*

PARALLELOGRAMS

The parallelogram has occupied from the beginning a prominent position in the theory of linkages [15, 16]. Its use as a basis for the familiar pantograph is illustrated in Figure 10. A rhombus *MANP′* has two legs extended to points *O* and *P* so that *O*, *P′*, and *P* are collinear. Triangles *OMP′* and *OAP* are always similar and thus

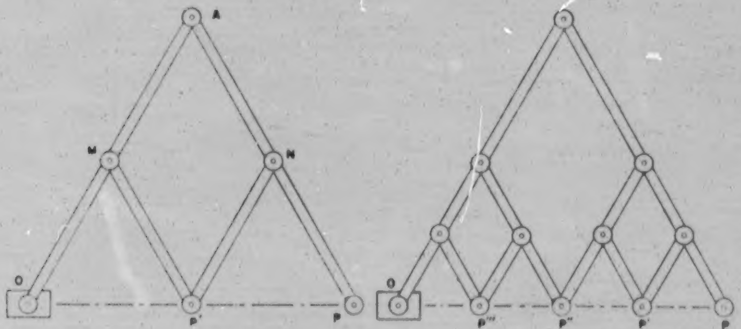

Figure 10

$$OM/OA = OP'/OP = 1/2,$$

or
$$OP = 2(OP').$$

Accordingly, if O is fixed and P be moved on some curve, the point P' traces a curve similar and similarly placed to the first but reduced in size by one-half.

An obvious extension of the pantograph to one with multiple copying points is shown also in Figure 10.

Clearly, the rhombus may be replaced by a general parallelogram if O, P', and P are kept collinear. This leads to the general pantograph shown in Figure 11, I. The bar MP' has been replaced by another bar OB without changing the effectiveness of the original parallelogram. The point P' on BN which is collinear with O and P is the copying point. Variously selected points P on the bar AN extended will produce an assortment of reduction factors determined by:

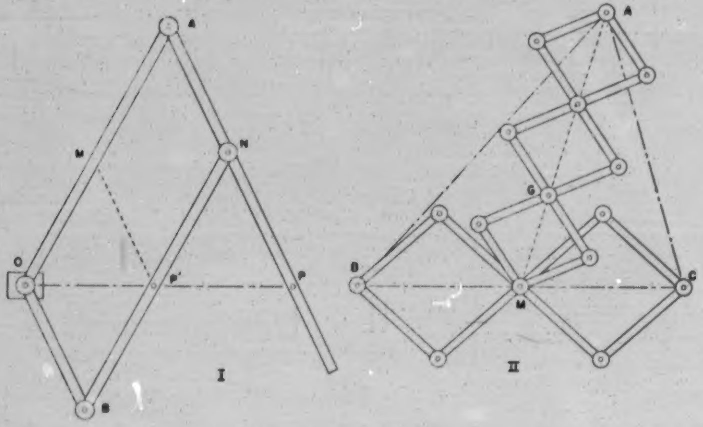

Figure 11

$$OM/OA = OP'/OP,$$

or

$$OP' = [(OM)/(OA)] \cdot (OP).$$

The linkage of Figure 11 II is offered for its novelty [14, 15]. Five rhombuses are joined together as shown. In all positions, M is the mid-point of BC, while G is the lower trisecting point of the line segment AM. Thus G is always the *centroid* of the variable triangle ABC.

Of considerable importance to linkage theory and the description of higher plane curves (including the conics) is the crossed parallelogram, already introduced here in Figure 4 [1, 2, 3, 4, 8, 12, 15]. Further study may be had through the listings in the attached bibliography.

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GRAPHICAL REPRESENTATION OF COMPLEX ROOTS

Howard F. Fehr

COMPLEX NUMBERS usually are first introduced into high school mathematics under the topic of Quadratics. The solution of quadratic equations by the methods of completing the square and substituting in the quadratic formula frequently leads to square roots of negative quantities. In order to admit expressions involving such quantities as roots of an equation, it becomes necessary to extend the number system to include complex numbers. The usual treatment of such numbers in the high school is mechanical and meaningless, but the Argand diagram can and should be used to represent such numbers as points in a plane.

The study of the quadratic function can well begin with the construction and study of its graph. In this way, it is easy to see that one of three conditions prevail: the x-axis cuts the graph at two distinct points; or it is tangent at one point; or there is no intersection. In the last case the student is told that the roots of $f(x) = 0$ are imaginary; frequently he uses this word in its ordinary literal sense and thinks of the roots as purely fictional. It would be better to use the word "complex" instead of "imaginary." It is the purpose of this article to show that the actual graphical location of these complex roots is within the comprehension of the average high school junior and senior.

Consider the function $y = x^2 - 4x + 8$. The graph of this function lies entirely above the x-axis. Hence the roots of $f(x) = 0$ are complex. From the graph it is evident that the roots of $x^2 - 4x + 8 = 4$ $(y = 4)$ are equal; if $y > 4$, there will always be two real roots. For the high school student who has not studied complex numbers, the equation $x^2 - 4x + 8 = 0$ has *no* roots, while the equation $x^2 - 4x + 8 = 6$ has exactly two roots. If he solves the first equation, he finds the answers to be $x = 2 \pm 2i$, but these values do not appear on the usual graph.

A common method of constructing these complex roots makes use of tangents to the graph. Thus in Figure 1, let the axis of the parabola meet the x-axis at A. From A draw tangents AR' and AR; then draw RR'. Let RR' meet the parabola's axis at Q. Then lay off on both sides of the x-axis $AP_1 = AP_2 = RQ$. Considering YOX as a complex plane, P_1 and P_2, then represent the complex roots of the equation.

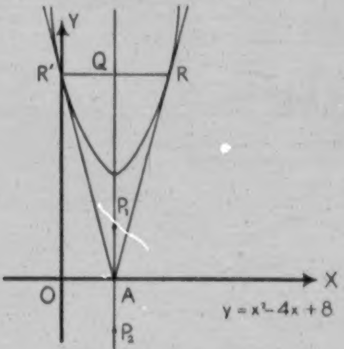

Figure 1

There are two principal objections to the use of such a construction in the high school course: first, the dual use of the plane as rectangular coordinates to represent functions of a real variable and also as a complex number plane is confusing and likely to lead to erroneous concepts; second, the proof that P_1 and P_2 represent the complex roots requires more advanced mathematics (such as a knowledge of slope, equations of tangent lines to a parabola from an external point, and some rather involved algebraic manipulations) than can be expected of the high school pupil. This method can be extended, however, to cubics and higher equations, and the interested reader can find more information in the article by Frumveller listed as a reference on page 138.

Most teachers will find the following method better suited to high school algebra.

We shall construct the graph of $x^2 - 4x + 8$ with both its real and its complex branches. In order to do this, we consider only real values of the function but both real and complex values of the variable x, as shown in the table below. Designate the function by y and solve for x.

y	x
8	2 ± 2
5	2 ± 1
4	2 ± 0
3	$2 \pm i$
0	$2 \pm 2i$
—5	$2 \pm 3i$

$$y = x^2 - 4x + 8$$
$$x^2 - 4x + 4 = y - 4$$
$$x = 2 \pm \sqrt{y - 4}$$

It is evident in this form that:

(a) There are two values of x for each value of y.

(b) If $y < 4$, the values of x are complex.

To construct the graph, we use the horizontal plane as the complex number plane in which values of the variable x are located. Through the point of origin of this plane we draw the y-axis per-

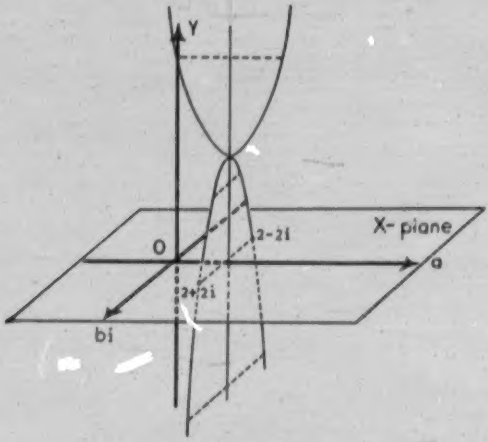

Figure 2

pendicular to the plane. Then for each value of y (as given in the table) we locate the corresponding two values of x. For $y > 4$ the points thus plotted give the ordinary parabola, which we call the real branch. For $y < 4$, the points fall in a plane which is perpendicular to the YOa plane and two units from the point of origin. These points form an inverted parabola which we call the complex branch. It is readily seen that this latter branch cuts the x-complex-number plane in the two points $2 \pm 2i$, which are the roots of the given equation. These points are actual intersections, as shown in Figure 2, and need not be imagined.

The construction can be attacked more generally by letting x have the complex value $a + bi$, and finding the conditions under which y is real. Thus if $y = x^2 - 4x + 8$, we have

$$y = (a + bi)^2 - 4(a + bi) + 8 = a^2 - b^2 - 4a + 8 + i(2ab - 4b).$$

If y is real, the coefficient of i is zero; hence

$$2ab - 4b = 0, \text{ and } b = 0 \text{ or } a = 2.$$

If $b = 0$, then $y = a^2 - 4a + 8$, which is the real branch of the quadratic drawn in the YOa plane.

If $a = 2$, $y = 4 - b^2$. This is the complex branch of the quadratic in the plane $a = 2$, which is parallel to the YOb plane. The complete graph, therefore, consists of two branches; and for any real value of y, the graph will always have two and only two corresponding values of x, either real, equal, or complex.

A permanent model can be made by using three glass panes, cemented together so as to form the YOa plane, the aOb plane, and the plane of the complex branch $a = 2$. The y, Oa, and Ob axes can be painted on the respective planes in black. The parabolas can then be painted on their planes, preferably in contrasting colors. Thus the real branch $a^2 - 4a + 8$ can be painted in blue on the YOa plane, while the complex branch $4 - b^2$ can be painted in red on the plane $a = 2$. The intersections $2 \pm 2i$ can easily be labeled in the x-plane.

Each student can make his own model from ordinary graph paper. First, fold the sheet in half, lengthwise. Draw the OY and Oa axes on both sides of the folded paper so that they correspond. Next, locate the axis of the parabola which is always at $x =$

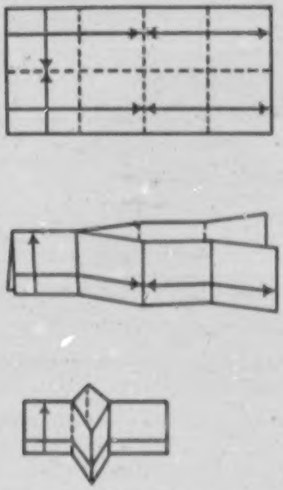

Figure 3

— $b/2a$, and draw this axis to correspond on both sides of the folded paper. Slit the lengthwise crease from the right edge as far as the parabola axis and fold each slit part into three equal sections, making the middle one bend outward. (See Fig. 3.) Then, by holding the slit ends together and pressing from both ends so that the middle sections bend outward, the paper is divided into two perpendicular planes, the YOa plane and the complex branch plane. Label the axes accordingly and plot the graph of each branch on all visible faces of the paper. Then looking directly toward the YOa plane, one sees only the real branch of the parabola; it does not intersect the x-axis. Turning the model 90° around the y-axis, one sees only the complex branch of the parabola, which intersects the bi axis at $\pm 2i$. Hence the roots are $2 \pm 2i$. When the entire graph paper is unfolded to a plane, it appears as in Figure 4.

To graph the general quadratic, we proceed in the same manner. Thus we substitute $a + bi$ for x in

$$y = px^2 + qx + r.$$

$$\begin{aligned} y &= p(a + bi)^2 + q(a + bi) + r \\ &= pa^2 - pb^2 + qa + r + bi(2ap + q). \end{aligned}$$

If y is real, the coefficient of i is zero; hence either

$$b = 0 \quad \text{or} \quad a = -q/2p.$$

If $b = 0$, $y = pa^2 + qa + r$ is the real branch.

If $a = -q/2p$, $y = \frac{-q^2}{4p} - pb^2 + r$ is the complex branch.

The cubic equation can be treated in a similar manner. In this case there are two complex branches which are located on the two

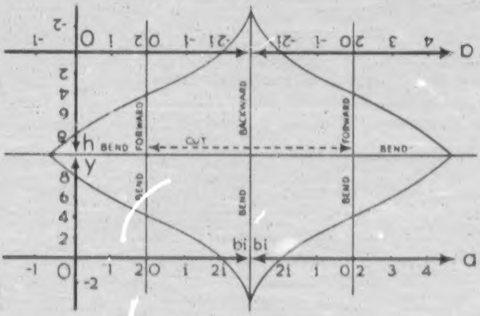

Figure 4

branches of a hyperbolic cylinder whose elements are perpendicular to the x-plane. This is best illustrated by a particular equation. Let $y = x^3 - x + 6$. As before, let y be real and x complex, and substitute $a + bi$ for x.

$$\begin{aligned} y &= (a + bi)^3 - (a + bi) + 6 \\ &= a^3 - 3ab^2 - a + 6 + bi(3a^2 - b^2 - 1). \end{aligned}$$

If y is real, the coefficient of i is zero; hence

$$b = 0 \quad \text{or} \quad 3a^2 - b^2 - 1 = 0.$$

If $b = 0$, $y = a^3 - a + 6$. This is the real branch of the cubic equation, which is the part usually given in the YOa plane.

If $3a^2 - b^2 - 1 = 0$, $y = -8a^3 + 2a + 6$, which formula gives the values of the function on the complex branch. Notice that $3a^2 - b^2 = 1$ is the equation of a hyperbola in the aOb plane. Hence any complex number $a + bi$ that satisfies $3a^2 - b^2 = 1$ will give the real value $-8a^3 + 2a + 6$ to y. That is, any complex number on the hyperbola in the aOb plane substituted in the original equation will produce a real value for y, which is plotted above the complex value of x. The fact that the values of y for the complex branches are given in terms of a only makes it easier to evaluate and locate the points on the graph. Thus if $a = 1$, $y = -8 + 2 + 6 = 0$; and the points on the hyperbola are $b^2 = 2$ or $b = \pm \sqrt{2}$. Hence the complex roots are $1 \pm i\sqrt{2}$. Similarly if $a = 2$, $y = -54$, and $b = \pm \sqrt{11}$. When $x = 2 \pm i\sqrt{11}$, the value of the function on the complex branch is -54.

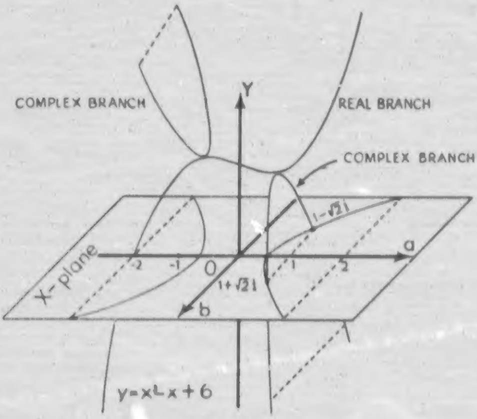

Figure 5

Figure 5 shows how the complex branches begin at the maximum and minimum points of the real branch and have all their points directly above or below the hyperbolas in the aOb plane; that is, they lie on the hyperbolic cylinders. Such a graph shows immediately the values of y for which the cubic equation has three real roots, two equal roots, or only one real and two complex roots. It clearly shows that a cubic equation always has three and only three roots.

The extension to the quartic equation should now be obvious. The quartic function has three complex branches. Thus if $y = x^4$, substituting $a + bi$ for x gives

$$y = (a + bi)^4 = a^4 - 6a^2b^2 + b^4 + 4abi(a^2 - b^2).$$

If y is to be real, either $a = 0$, $b = 0$, or $a^2 - b^2 = 0$.

If $b = 0, y = a^4$, which is the real branch.

If $a = 0, y = b^4$, which is the normal complex branch.

$$\left.\begin{array}{l}\text{If } a = b, y = -4a^4,\\ \text{If } a = -b, y = -4b^4,\end{array}\right\}$$ which are the two remaining complex branches. Thus a plane $y = k$ cuts the graph of $y = x^4$ always in four and only four points. If k is positive, there are two real and two complex roots; if k is zero, there are four equal roots; and if k is negative, there are four complex points of intersection.

With a slight change in method, the complex intersections of

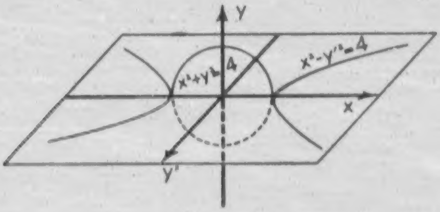

Figure 6

some conics can be shown as in Figure 6 above. In the circle $x^2 + y^2 = 4$, we notice that if $x > 2$, $y = \pm\sqrt{4 - x^2}$ is a pure imaginary number. If we agree to give only real values to x, then for $x > 2$ and $x < -2$ we plot the pure imaginary values of y in the horizontal plane, since the real values were in the vertical plane. This is equivalent to substituting iy' for y in the equation of the circle from which we obtain $x^2 + (iy')^2 = 4$ or $x^2 - y'^2 = 4$. Hence the complex branch of the circle for real values of x is a hyperbola which is plotted in the horizontal plane.

The following problem illustrates how the complex intersections of two quadratics can be shown by using the above method of graphing the complex branches. Let it be required to solve the set of equations $x^2 + y^2 = 25$ and $3y^2 = 16\,x$. By ordinary algebra we find that the solutions are $x = 3$, $y = \pm 4$ and $x = \frac{-25}{3}$ $y = \frac{\pm 20i}{3}$. The ordinary graphs of the real branches of the circle and parabola show the intersections (3, 4) and (3, —4). If we draw the complex branches, the other solutions will be apparent. The complete graph is shown in Figure 7.

The complex branch of the circle is the hyperbola $x^2 - y^2 = 25$. To obtain the complex branch of $3y^2 = 16x$, x real, let $y = a + bi$. Then $3(a^2 - b^2 + 2abi) = 16x$. For x real, the coefficient of i is zero and $a = 0$ or $b = 0$. If $b = 0$, $3a^2 = 16x$, which is the real branch. If $a = 0$, $-3b^2 = 16x$. This is the complex branch which is plotted in the horizontal plane. The intersection of the hyperbola and the complex-branch parabola gives the complex solutions.

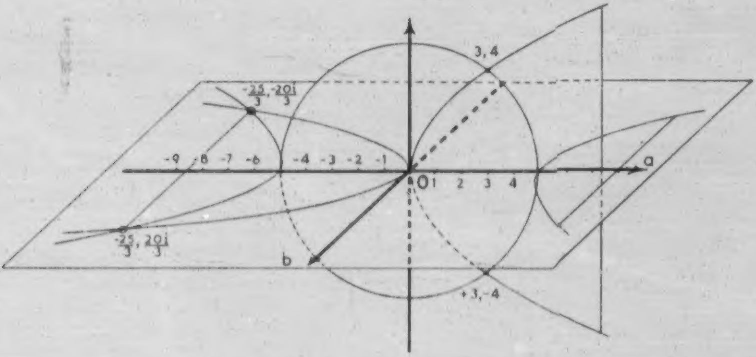

Figure 7

It is evident that the complex intersections of a straight line with a conic cannot be shown in this way, since in a linear function if x is complex so is y, and conversely. There is no real value for a linear function if the variable is complex.

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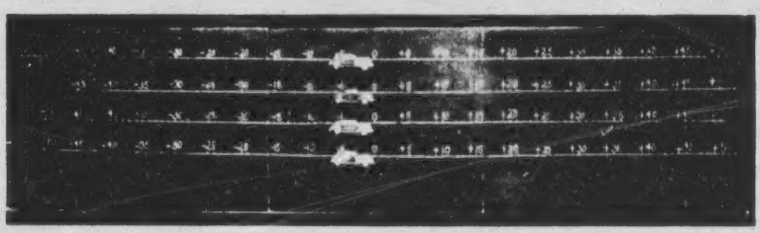
A DEVICE FOR TEACHING THE FUNDAMENTAL OPERATIONS WITH DIRECTED NUMBERS

Hortense Rogers



About 1925 an automobile race track was built near the highway between Rock Hill, South Carolina, and Charlotte, North Carolina. People from far and near attended the races held there. One day a student-teacher of mathematics in the Winthrop Training School brought to class a poster with a race track on it. This poster suggested to the supervisor the idea of using a race track for teaching the operations with signed numbers. The results showed real interest and a better understanding on the part of the students, and so the device has been used, with variations, for eighteen years. The boys and girls enjoy the race, and directed numbers take on a real meaning for them. The operations are rationalized through their concrete experiences; the examples used are literally made by the students. Furthermore, from a study of these examples they make their own algebraic rules which are later stated in good form. They do not question the rules for using signs, for to them those rules are reasonable. Although it takes a little longer to teach addition of directed numbers in this way, the thinking done and the understanding acquired by the students justify the slower progress made in the beginning of the study.

A clear understanding of certain concepts should precede the study of the operations with directed or signed numbers. These concepts include: directed or signed numbers, negative number, positive number, the sign of a number, the absolute value, the reading and writing of such numbers, the number scale, directions on the scale, the meaning of a number negatively directed or positively directed with respect to another number, and so on. As each operation is introduced, a definition or meaning of the operation should be given to and understood by the students. The meaning of addition used with this device is in keeping with that given by Breslich and Everett.[1]

The teacher may introduce the idea of having a race in any way he wishes. The students will soon be enthusiastic and, with some guidance, will plan the race themselves. The following account may make it appear that the idea, carefully detailed, is handed out to the students. This is far from being the case. They really do discuss, plan, and make decisions. They will discuss the different kinds of races; and, after deciding on an automobile race, will talk about teams, tracks, cars, distances, and provision for these in the classroom. They will divide into teams and will appoint at least three committees: automobile, track, and planning. These committees will be working simultaneously and will be ready by a set time. Each committee will need some guidance.

The automobile committee. Each team has an automobile committee. This team decides that the cars must face to the right, the positive direction. All members look in magazines or newspapers for pictures of cars. Each committee selects from those submitted the one its team will use. They mount the picture on cardboard; then they cut it out, carefully following the outline of the car. Next, near the top of the car they punch two holes through which a string on the race track is to be run. If convenient, these holes can be strengthened with nickel brads. The cars are now ready and are turned over to the track builders.

The track committee. This committee is composed of one or two members from each team. They usually select front blackboards for the race track and use as much space as possible. They fasten small strips of wood across the extreme ends of the blackboard. In each of these strips they place as many staples as there are teams. Then they tie strong twine to the staples at one end, run on the automobiles, and tie the twine to the corresponding staples at the other end of the blackboard. Now, after placing all automobiles in the middle of the strings, they draw clear lines just behind the tracks. They do this more easily if they use a yardstick. By starting at the middle of the line, they measure off accurately spaces one and one-half inches wide, marking each with a very short vertical line, but making each fifth line a little longer than the others. The marks appear as straight mileposts along the tracks. It is well to select as a marker of the posts a pupil who can make good figures. Starting at zero, he puts the minus and plus numbers above the fifth posts only. The committee's job at this point is completed.

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[1] E. R. Breslich, *Problems in Teaching Secondary School Mathematics*, p. 120. Chicago University Press, 1931. Also, J. P. Everett, *The Fundamental Skills of Algebra*, p. 45. Bureau of Publications, Teachers College, Columbia University, 1928.

The planning committee. This committee, which may be composed of a few from each team or of the whole class, decides on all other questions. They may decide on a relay race so that each pupil may actually drive a car. The order of driving should be agreed upon to suit the group. The next question is how far each shall drive. Games in which spinners are used are recalled, and they decide on a similar device, but with the additional specification of having both negative and positive numbers. Each car will travel the number of points indicated on the counter. The spinners we use in Winthrop Training School were bought at the dime store. The [illegible] were added, and they now carry numbers from minus eleven to plus twelve. A good substitute for a spinner for each team is a covered box with many slips of paper each bearing a small signed number. Each player draws a slip and moves his car the distance specified thereon. If all teams are to draw from the same box, there should be in the box at least ten of each number used.

Definite instructions for each driver are agreed upon as follows:

(a) He tells where the car is before he starts.
(b) He spins, and tells the class what he gets. He also states in what direction the sign tells him to go and how far the absolute value tells him to go.
(c) He goes to the race track and, after actually counting the number of spaces in the correct direction, he moves his car.
(d) He tells where his car is then.

Of course, all games require a referee-scorekeeper; usually the teacher serves. The following are his duties:

(a) He writes fairly large and spaces the scores well.
(b) He writes the name of each team on a board. If there is a lack of board space, he can use paper and before the next class session make mimeographed copies of the scores to give to the students. The use of the board is preferable for the class discussion.
(c) He writes for each driver the following:
  (1) The number showing where his car is when he starts.
  (2) Under that, the number he gets when he spins.
  (3) The number showing where his car is after he moves it the driving distance.

A study of the sample score board that follows will make clear the method of keeping the scores.

The students select the *name* for their own team. They usually decide that the winner shall be that team whose car is farthest to the right after a specified number of players on all teams have driven.

These preliminary arrangements having been completed in class or out of class hours, the group is ready for the race. They first have a trial heat in which only one on each team drives. The race is on! The boys and girls enjoy the activity; they have lots of fun and much freedom. Such comments as "put on your brakes," "don't go in reverse," some groans, some applause, and so on, are in order. As an award they put blue, red, or yellow asterisks on the winning cars. Invariably they request to race again the next day. This privilege is sometimes, but not always, granted.

Of course, the time spent on such an activity must be used for real teaching and real thinking. The adding of signed numbers is kept in mind while the students are counting the distances and moving the cars. They are quite willing to accept the fact that the results are the correct sums. But they cannot always have cars and tracks. It is now the teacher's task to guide the class in using these examples which they have made themselves to discover how to add directed numbers without the aid of the number scale. Toward that end then study the score board, a sample of which follows.

SCORE BOARD

Silver Streak

0	− 8	− 4	−13	−13	− 4
− 8	+ 4	− 9	0	+ 9	+ 7
− 8 D	− 4 Ⓒ	−13 B	−13 D	− 4 Ⓒ	+ 3 C
+ 3	+ 8	+ 6	0	+ 2	+ 4
+ 5	− 2	− 6	+ 2	+ 2	+10
+ 8 A	+ 6 C	+ 0 E	+ 2 D	+ 4 A	+14 A

Lucky Thirteen

0	− 5	+ 1	− 9	−12	−12
− 5	+ 6	−10	− 3	0	+10
− 5 D	+ 1 C	− 9 Ⓒ	−12 B	−12 D	− 2 Ⓒ
− 2	+ 5	+ 8	+ 6	− 3	− 4
+ 7	+ 3	− 2	− 9	− 1	+ 4
+ 5 C	+ 8 A	+ 6 C	− 3 Ⓒ	− 4 B	0 E

Spitfire

0	− 9	−10	− 2	+ 3	+ 7
− 9	− 1	+ 8	+ 5	+ 4	−10
− 9 D	−10 B	− 2 C	+ 3 C	+ 7 A	− 3 Ⓒ
− 3	− 8	+ 1	+ 4	− 6	−16
− 5	+ 9	+ 3	−10	−10	− 8
− 8 B	+ 1 C	+ 4 A	− 6 Ⓒ	−16 B	−24 B

As an aid in studying the combinations on the score board the students mark, using capital letters, groups of examples having certain combinations. First, they find all the sums when each addend is positive. They mark these with an A and study them. These observations enable them to discover the answers to the following questions:

(a) What is the sign of the result?
(b) Is this the sign of each one marked?
(c) What is the absolute value of this result?
(d) How can one get this absolute value from the absolute values of the two given numbers? It may be necessary to suggest the answer by asking which one of the four operations one can use.
(e) Is this true of the absolute value of each one marked?

The class is then guided in stating all this in one good clear-cut sentence. Then the teacher writes on the board:

> In adding two positive numbers, use the ——— sign and ——— the absolute values.

The students should read this together several times, filling in the blanks with the words *plus* and *add*. Later it would be well for one child to go to the board and write the words above the blanks.

Now have the students mark with a B all sums made from two negative numbers. Follow the same procedure as above, asking similar questions and using a similar sentence.

Next, have them mark with a C all sums made from a negative and a positive number. Ask the same first question. Also, ask whether this is true of all marked with a C. The answer is no. Hence, have them draw a circle around each C alongside the answers that are negative and leave only a C for all that are positive. Again have them study the two groups for a clue as to how to decide on the sign. Although this procedure will take more time than was required for the preceding groups, nearly all the students will discover and agree that the sign is the sign of the one having the larger absolute value. Then they can study these numbers to find how to get the absolute value. The sentence which the group formulates will then be written on the board:

> In adding a negative and a positive number, use the sign of the number having the ——— absolute value and ——— the smaller absolute value from the greater absolute value. (The correct words here are respectively *greater* and *subtract.*)

Similarly, they mark with a D and study the examples with one number and a zero. The sentence will be:

> In adding zero and either a negative or a positive number, the answer is the ——— negative or positive number. (The correct word is *given.*)

Similarly, they mark with an E and study the examples with the answer zero. The sentence will be:

> In adding a negative and a positive number with the same absolute value, the answer is ———.

What other combination gives zero? The answer is "two zeros." This step may be omitted.

These generalizations should be followed immediately by practice in applying each rule discovered and stated by the class. Some examples worked by applying the rules should be checked by using an automobile. The top track and car are usually left on the board for several weeks. This plan keeps a number-scale before the class, but permits the use of the board for other purposes.

The automobile race, with necessary variations, has been used to teach subtraction and multiplication of directed numbers also. The interest of a given group should determine how long to continue its use. It is a device that has proved profitable for many students and it probably will clarify for others a basic matter that has often proved a stumbling-block.

TRISECTION

Robert C. Yates

THE PROBLEM of trisecting the angle has been a fertile source of mathematical ideas and to it much of the history of mathematics owes its origin. For some time it has been known that the general angle cannot be trisected by straightedge and compasses [10]; yet many still entertain a hope that the impossible may be accomplished [15]. Certain angles, notably 90°, 54°, 9°, do admit of trisection by straightedge and compasses. Some geometrical tools, a few of which are to be discussed here, are capable of solving this problem. There are curves,* other than the straight line and circle, which present means by which the general angle may be trisected [15]. Outstanding among these are the quadratrix, the conchoid, the trisectrix (a special limacon), and the parabola.

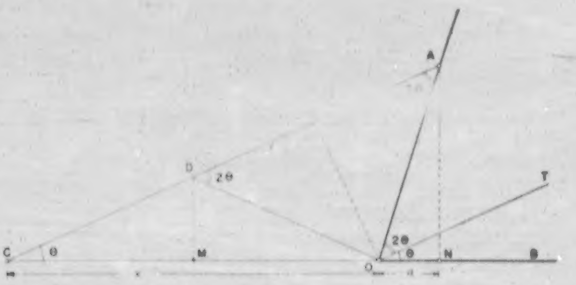

Figure 1

To understand the problem and to establish the impossibility of straightedge-compass trisection in the general case, it is necessary to form an interpretation in the field of analysis. Let the $\angle AOB = 3\theta$ (Fig. 1), and suppose one of the trisecting lines to be OT, so that $\angle TOB = \theta$. Select an arbitrary length on OA as the unit distance and draw the parallel AC to OT, meeting OB

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* An infinitude, in fact.

extended at C. Then $\angle DCO = \theta$. Now draw OD equal to the unit length so that triangle AOD is isosceles with base angles 2θ (since they are alternate interior angles, $\angle DAO = \angle AOT$). It is evident, since $\angle ADO$ is the sum of the opposite interior angles of triangle DCO and $\angle DCO = \theta$, that $\angle DOC = \theta$. Thus triangle DCO is isosceles and $DC = DO = 1$. Let x denote the distance OC, $2y$ the distance AD, and a the projection of OA upon the side OB (i.e., $\cos 3\theta$). From the similar right triangles CMD, CNA, and CLO, we have:

$$x/2 = (x + a)/(1 + 2y) = (1 + y)/x.$$

The elimination of y here produces:

$$x^3 - 3x - 2a = 0.$$

Since we may construct the quantity $a = \cos 3\theta$, we may think of this as being given with the angle AOB. If the point C, or its distance x along OB, can be determined, the problem is at once solved by connecting C to A and then constructing the trisecting parallel OT. Thus the geometrical solution of the problem is equivalent to the algebraic solution of the corresponding trisection equation.

This cubic equation cannot in general be decomposed into factors with real constructible coefficients [5, 15] and thus its roots cannot be obtained by straightedge and compasses.

The carpenter's square [12, 13] is essentially a "T" formed by

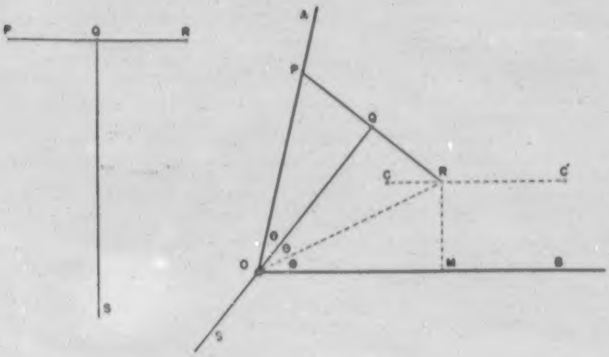

Figure 2

joining rigidly together a segment and its perpendicular bisector. Let $\angle AOB$ (see Fig. 2) be trisected. Place the edge QS of the "T" along one side OB of the angle and mark two distinct points, C and C', at a distance PQ $(= a)$ from OB. Draw CC'. Insert the segment PR so that P falls upon OA, R on CC', with the edge QS passing through O. Then $\angle AOB$ is trisected by QS, for, from the congruent right triangles POQ, ROQ, ROM,

$$\angle POQ = \angle ROQ = \angle ROB.$$

The tomahawk [6, 15] is similar to the carpenter's square. It is formed by rigidly attaching a tangent line to a semicircular disk whose diameter is extended the length of the radius (see Fig. 3).

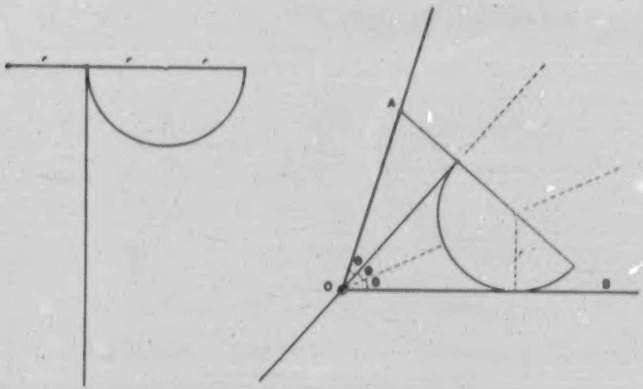

Figure 3

Its application to the trisection of a given angle AOB is immediate.

The two foregoing instruments may also be used to draw the cissoid and duplicate the cube (i.e., extract the cube root of a selected segment) [17].

A single straightedge carrying two arbitrarily placed marks, P and Q, will trisect the general angle if assisted by the compasses [3, 8, 15, 17]. Let the distance PQ be $2a$. Upon one side OB of the given angle establish $OT = a$. Draw TX and TY perpendicular and parallel respectively to OA. Place the ruler through O so that P and Q fall on these lines as shown. Then OPQ trisects $\angle AOB$, as shown in Figure 4, page 149. For, if M be the mid-point of PQ,

$$PM = MQ = MT = a$$

and
$$\angle MQT = \angle MTQ.$$

Since $\angle OMT$ is an exterior angle of triangle MQT, and further since $MT = OT = a$, then

$$\angle MOT = \angle OMT = 2\,(\angle MQT).$$

But since OQ traverses two parallel lines,

$$\angle AOQ = \angle OQT.$$

Thus

$$\angle MOT = 2\,(\angle AOQ)$$

and OQ is a trisector of $\angle AOB$.

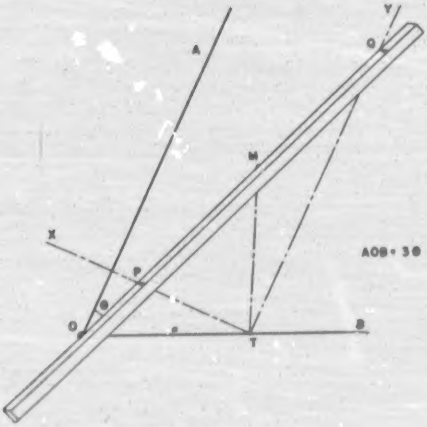

Figure 4

It should be noted that as P travels along TX with the straightedge always through O, the path of Q is one branch of the conchoid of Nicomedes, one of the curves prominent in the history of trisection [15, 17].

As P and Q travel upon the fixed lines TX and TY, the path of any selected point of PQ is an ellipse having the fixed lines as axes of symmetry. The envelope of the lines PQ, and of the family of ellipses thus formed, is the astroid, the four-cusped member of the hypocycloid family [16, 17].

The paper and cone [1, 15] combination affords an interesting trisection arrangement. A right circular cone is constructed, having its slant height equal to three times the base radius. The cone is placed so that the center of the base is coincident with the vertex of the given angle $AOB = 3\theta$. Then arc $AB = 3r\theta$. A sheet of paper is now wrapped around the cone, and the points A, B, and V are marked on it as shown. When the sheet is removed and flattened out, the angle AVB is one-third the angle AOB, as shown in Figure 5, above. For, since $AV = 3r$,

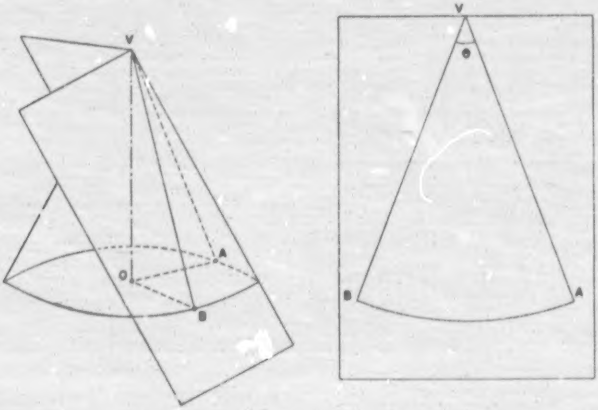

Figure 5

$$\text{arc } AB = 3r\,(\angle AVB)$$

and thus

$$\angle AVB = \theta = (\angle AOB)/3.$$

The *instrument of Pascal* [1, 15, 16] is composed of three bars and incorporates a straight groove, or slot, in which a point F slides. The two bars, OE and OF, are taken equal in length and hinged together at O. The grooved bar ED is extended to C so that $CE = OE = OF = a$. To trisect a given angle AOB, fix the bar OF upon a side OB of the angle and move the point C until it falls on the extension of OA, as shown in Figure 6. Now, since $\angle OCE = \angle COE = \theta$ and $\angle OEF = \angle OFE = 2\theta$,

$$\angle AOF = 3\theta.$$

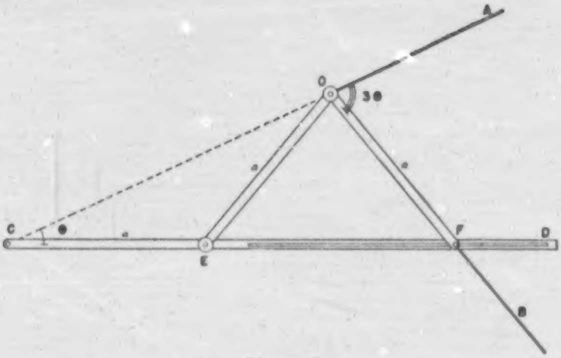

Figure 6

Other features of this mechanism are noteworthy. If the bar *OF* is fixed, as is the case in trisecting an angle, the path of *C* is a limacon of Pascal. If, instead, *CD* is fixed, any selected point of *OF* (or any point rigidly attached to *OF*) describes an ellipse symmetrical about *CD*. If *C* is fixed and *O* be moved along a fixed line *CA*, then *F* describes the cycloidum anomalarum of Ceva, a curve resembling the lemniscate.

The common pantograph [4, 9, 15, 16] is formed from a hinged parallelogram with extended legs. If the Pascal device of Figure 6

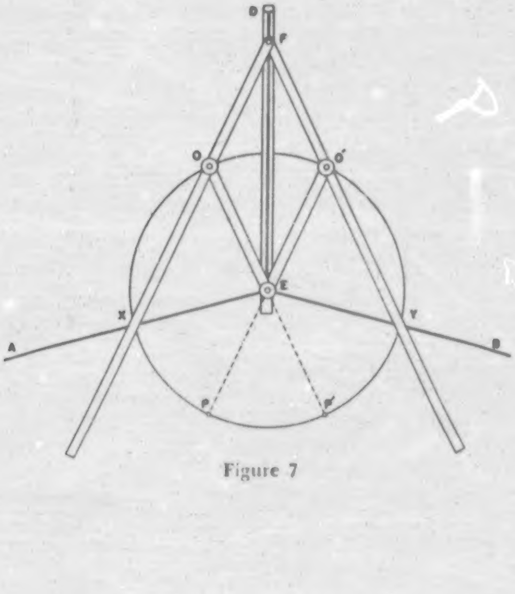

Figure 7

is reflected in the bar CD, the result (if the bars OF and $O'F$ are extended) is the apparatus shown in Figure 7. Let X and Y be the points of intersection of the sides of a given angle AEB, with the circle having E for center and EO for radius. Fix the diagonal groove along the bisector of the given angle and let the point F move until the bars OF and $O'F$ pass through X and Y. Then $\angle OFO' = (\angle AEB)/3$, for it is obvious that

$$\text{arc } XP = \text{arc } OO' = \text{arc } P'Y$$

and since arc OO' = arc PP',

$$\angle XEP = \angle PEP' = \angle P'EY = \angle OFO'.$$

The grooved bar may be discarded, of course, if some other arrangement is made to guide the point F along the bisector.

Crossed parallelograms [8, 15] form a trisector. Let $OABC$ be constructed from four bars, equal in pairs and crossed. That is,

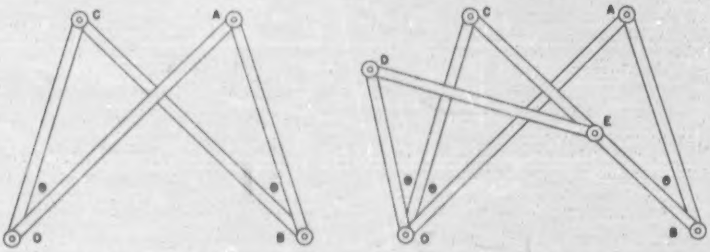

Figure 8

$OA = BC = a$; $OC = AB = b$, with the bars hinged at O, A, B, and C. It is evident that the angles at O and B remain equal through all deformations of the parallelogram. A second crossed parallelogram is joined to the first as shown, so that $DE = OC = b$ and $OD = CE = c$. Generally, $\angle DOC \neq \angle COA$. However, if these angles are to be equal, the two crossed parallelograms must be similar and have proportional sides, and conversely so. (See Fig. 8.) Thus, if

$$\angle DOC = \angle COA,$$

then
$$a/b = b/c \quad \text{or} \quad b^2 = ac.$$

If a third crossed parallelogram be joined to the second in similar fashion, three equal angles will be formed at O as shown in Figure 9. The four lengths involved form a geometric progression.

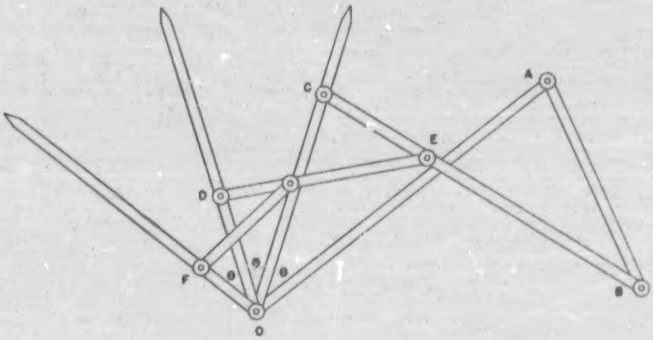

Figure 9

The bars OA, OC, OD, etc., may be extended to convenient lengths for the sake of appearance.

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PAPER FOLDING

Robert C. Yates

THE ART of paper folding has been transplanted into the field of plane geometry with fascinating results by the Indian, T. Sundara Row [12]. With a few assumptions, it is not difficult to show that *all Euclidean plane constructions may be effected by folding and creasing* [16].

It must be remarked here at the start that a crease in a sheet can be assumed straight not because two planes intersect in a line, but rather for the following reasons. Let P and Q be two points of the paper that are brought into coincidence by the folding process. Then any point A of the crease is equidistant from P and Q, since the lines AP and AQ are pressed into coincidence. Hence the crease, being the locus of such points A, is the perpendicular bisector of PQ [15, 16]. We make the three assumptions:

(a) that we may place one point of the sheet upon another and thus create a straight crease;
(b) that we may establish the crease through two distinct points;
(c) that we may place a given point upon a given line so that the resulting crease passes through a second given point.[1] This implies the ability to fold a crease over upon itself or upon another.

ELEMENTARY CONSTRUCTIONS

With these assumptions, the following exercises may be introduced into classroom activities. They are presented merely as suggestions. Interested persons will have no trouble in extending the list. A particularly fine medium for folding is the wax paper invariably found in kitchen cabinets.

(1) Establish the crease through a given point P perpendicular

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[1] We assume the point and line are so situated that this may be accomplished.

to a given line *L*. (Fold *L* over upon itself so that the crease passes through *P*.)

(2) Establish the crease through a given point *P* parallel to a given line *L*. (Obtain two perpendicular creases.)

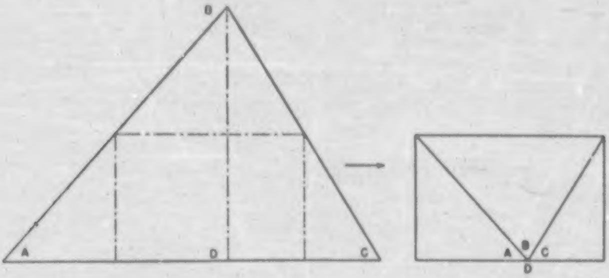

Figure 1

(3) Obtain the foot *D* of an altitude of a given triangle *ABC*. Now fold the vertices *A, B, C* over to *D*. This model illustrates the three basic and important theorems (Fig. 1):

(a) the sum of the angles of a triangle is 180°,
(b) the line joining mid-points of two sides of a triangle is parallel to and equal to half the third side; and
(c) the area of a triangle is half the product of a side and its altitude. A large model made from cardboard and Scotch tape will prove its worth.

(4) With a given triangle, fold the sides over upon themselves to obtain the angle bisectors and the incenter.

(5) With a given triangle, fold each vertex over upon the other two to obtain the perpendicular bisectors of the sides and the circumcenter.

(6) With a given triangle, fold each side over upon itself so that the crease passes through the opposite vertex. These creases are the altitudes which meet in the orthocenter. The triangle formed by the feet of the altitudes is the triangle of least perimeter that may be inscribed to a given triangle [2, 10, 14, 16].

(7) With a given triangle, establish the crease through each vertex and mid-point of opposite side. These creases are the medians which meet in the centroid.

(8) With a given square, fold the corners to the center. The creases form a second square whose area is half the first. This continued process illustrates the sequence

$$1/2,\ 1/4,\ 1/8,\ 1/16,\ \ldots\ 1/2^n,$$

which occurs frequently in every algebra course.

(9) With a given square, fold and crease the inscribed square. Find the intersections of creases bisecting the angles between the sides of inner and outer square. These intersections are vertices of a regular octagon (Fig. 2); or

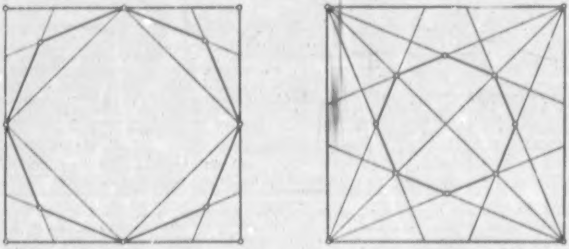

Figure 2

(10) Crease the quadrisectors of the angles of a given square and thus obtain the vertices of a regular octagon (Fig. 2).

The Conics

If the word "line" in assumption (c) be interpreted to mean "curve," the construction of the conic sections becomes permissible[2] [5, 7, 13].

Ellipse. For the ellipse (Fig. 3A) fold a point A, selected within a given circle of radius r and center O, over upon the circle as at X. Let the crease YZ meet OX at P. Then, since YZ is the perpendicular bisector of AX, $AP = PX$. Thus $OP + AP = OP + PX = OX = r$, a constant. Accordingly, the locus of P is the ellipse with O and A as foci. The crease YZ is tangent to the ellipse at P since $\angle APZ = \angle ZPX = \angle OPY$ insures the *optical property*. Special cases arise if A is taken on the given circle or at its center.

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[2] This implies the construction of all those curves known as "podoidal."

If the former, the locus is a single point—the center of the circle; if the latter, the locus is a circle.

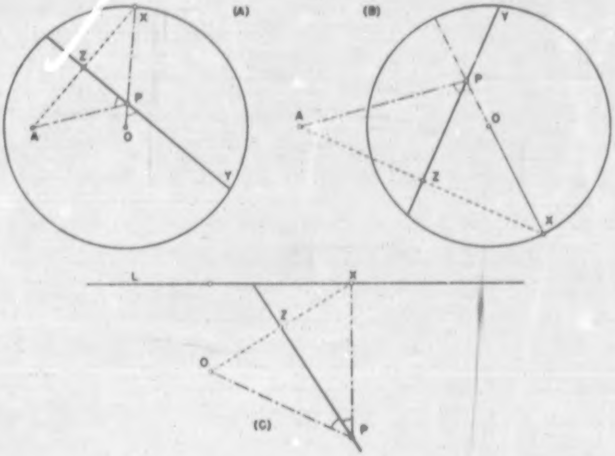

Figure 3

Hyperbola. For the hyperbola (Fig. 3B) the point A is selected outside the given circle. The locus of P, the point of intersection of the crease YZ and the extended radius OX, is the hyperbola having O and A as foci and all creases as tangents. For, since YZ is the perpendicular bisector of AX, $AP = PX$. Thus $AP - PO = PX - PO = OX = r$ and $\angle APZ = \angle ZPO$. The asymptotes are the creases determined by points X that are intersections of the given circle and the circle on OA as a diameter. The equilateral (rectangular) hyperbola results if $OA = r \cdot \sqrt{2}$.

Parabola. For the parabola (Fig. 3C) a point O is folded over upon a given line L as at X. The perpendicular to L from X meets the crease at P. The locus of such points P is the parabola having L as directrix, O as focus, and the creases as tangents. For, since PZ is the perpendicular bisector of OX, $OP = PX$ and $\angle OPZ = \angle ZPX$.

REGULAR POLYGONS

The foregoing examples represent only a part of the exercises possible by folding and creasing. The scope of this set can be considerably extended if we admit a process of "knotting" a paper strip whose edges are parallel [8, 9, 16]. For example, the regular polygons with an odd number of sides may be formed from a single strip (Fig. 4). For the pentagon, tie a single overhand knot, tighten carefully, and press flat. To form the heptagon (not constructible by straightedge and compasses), first tie an overhand knot as in the construction of the pentagon. Before tightening, however, pass the lead under and back through the knot.

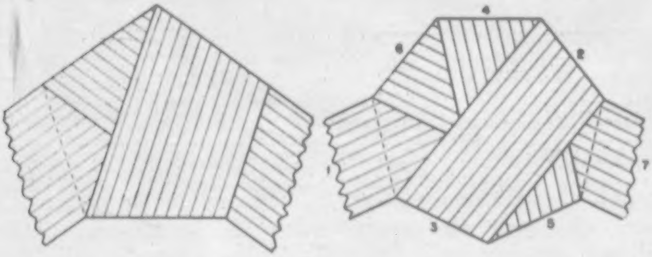

Figure 4

Regular polygons with an even number of sides may be formed similarly by means of two strips of the same width (Fig. 5). The

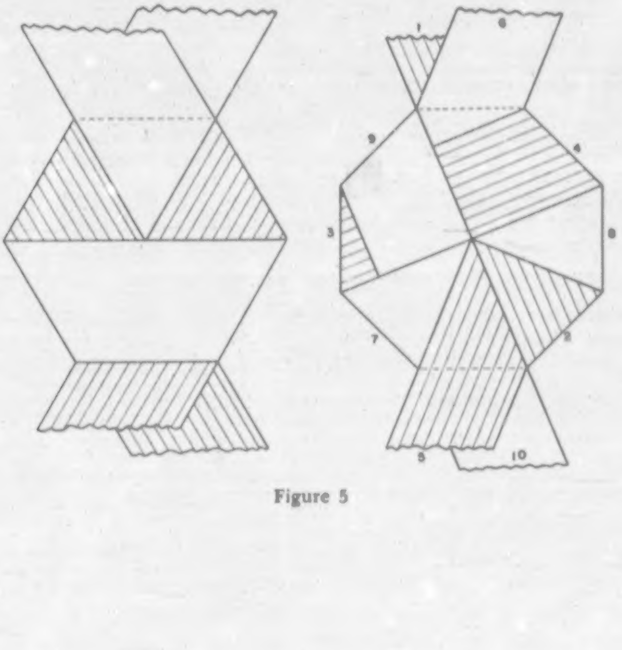

Figure 5

hexagon results from tying the sailor's reef or square knot (i.e., tuck the ends of each strip into a loop formed in the other). The formation of the octagon is very much more involved.

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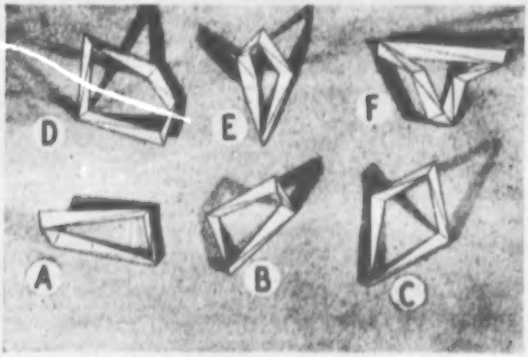
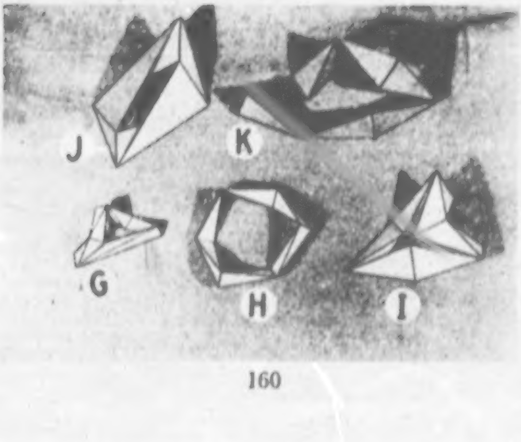
LINKAGES IN THREE DIMENSIONS

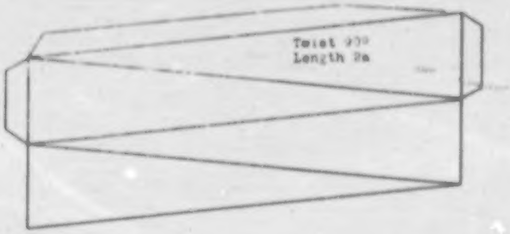
Michael Goldberg

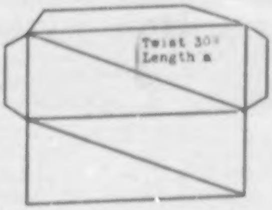
A CLOSED HINGED chain of seven or more links in space is always movable. For fewer than seven links, the linkage is generally rigid. Figure G is such a rigid five-bar linkage. However, special conditions may permit motion of a hinged linkage of six or fewer links. Several examples are included in the accompanying illustrations, and their specifications are listed on page 162.

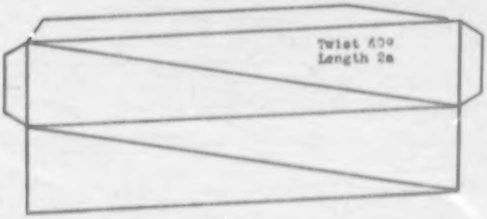




The links are pictured as rigid tetrahedra, each of which is hinged to its neighbors along two opposite edges (except for Figure J, in which the links are flat plates). The length of a link is the length of the common perpendicular between the hinges in a link. The twist of a link is the angle between the hinge axes in the link.







The developments of three tetrahedral links are shown. These may be used as templates to cut pieces of drawing paper which may be folded and pasted, by means of the flaps, to form the tetrahedra. These three forms suffice to make the linkages A, D, H, K, and C. The links may be joined by adhesive cloth, preferably on the inside of the links, where it is less likely to work loose.

Linkage A can be formed by using two links of the 90° twist and two links of the 30° twist. Linkage D can be formed by using

	Type of Linkage	*Twist and Length of Links*					
A	Bennett four-bar	90° $2a$	30° a	90° $2a$	30° a		
B	Bennett four-bar	90° $2a$	45° $\sqrt{2}a$	90° $2a$	45° $\sqrt{2}a$		
C	Bennett four-bar	60° a	120° a	60° a	120° a		
D	Goldberg five-bar	90° $2a$	60° $2a$	90° $2a$	30° a	30° a	
E	Goldberg five-bar	75° $(1+\sqrt{2})a$	90° $2a$	45° $\sqrt{2}a$	30° a	90° $2a$	
F	Goldberg six-bar	125° $(1+\sqrt{2}+2\sin 50°)a$	90° $2a$	50° $2a \sin 50°$	30° a	45° $\sqrt{2}a$	90° $2a$
G	Rigid five-bar	90° $4a$	90° $4a$	90° $4a$	90° $4a$	90° $\sqrt{97}a$	
H	Symmetric six-bar	90° a	90° a	90° a	90° a	90° a	90° a
I	Bricard rectangular six-bar $A^2+B^2+C^2=a^2+b^2+c^2$	90° A	90° a	90° B	90° b	90° C	90° c
J	Bricard octahedral six-bar		From movable octahedron				
K	Seven-bar		90° twists, equal length links				

two links of the 90° twist, two links of the 30° twist, and one link of the 60° twist. Linkage H can be formed by using six links of the 90° twist. Linkage K can be formed by using seven links of the 90° twist. Linkage C can be formed by using two links of the 60° twist and two links of 120° twist obtained by assembling the tetrahedron "inside-out."

The construction of linkage J is given in an article by the author noted below.

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THE PREPARATION AND USE OF NOMOGRAPHIC* CHARTS IN HIGH SCHOOL MATHEMATICS

Douglas P. Adams

THE SIMPLE alignment chart has proved a very attractive problem for elementary students. Some of them are delighted with its practical value; others are pleased by the simplicity of its operation; a few are interested only in its mathematical properties.

Plane geometry and logarithms are a sufficient foundation to carry the student far into the elementary aspects of the subject. A number of competent treatments of alignment charts have been written which require no more advanced mathematics than these high school subjects provide. The prime need for accurate drafting of the chart once it has been worked out frequently causes interest in the subject to arise in the school's drafting department. Whether, in general, the mathematics department or the drafting department is the better equipped to support this interest is an academic question obviously varied in answer and of small apparent value. High school students have a way of answering it by themselves.

The present article touches only one type of chart among several, all of which are quite simple. This type, however, is enormously valuable and is without question the one which should be thoroughly mastered before any others are investigated. If sufficient interest is shown, other types can be treated later.

Let us see how simple it is to develop an alignment chart. An ordinary square-ended trapezoid has been drawn in Figure 1A. In elementary school it is customary to turn this figure so that the two parallel sides are also parallel to the bottom of the sheet. These two sides are then called bases *A* and *B*, and we quickly show that if a third parallel *C* is drawn halfway between them, its length will be the average length of the two and will be given by the formula $C = (A + B) / 2$.

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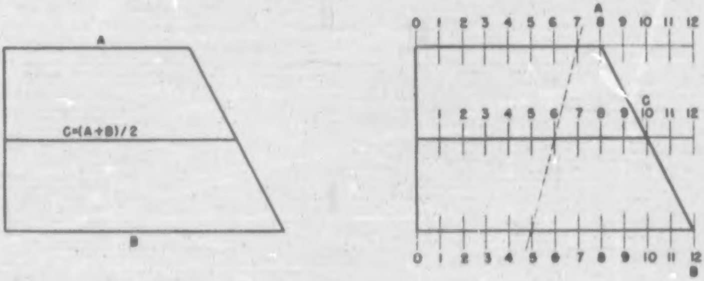
* An alignment chart is a particular type of nomographic chart. The general term "nomographic chart" is more widely known and the two are frequently, though improperly, used interchangeably. In this article only the alignment chart will be discussed.

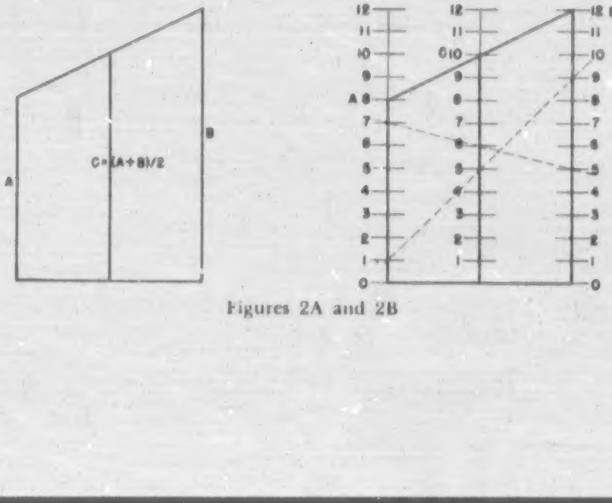
In Figure 1B, scales have been measured off on each of these three lines with zero points at the square end. Points *A* and *B* lie on these scales at the distances *A* and *B* from the square end.


A
C=(A+B)/2
B
A
0 1 2 3 4 5 6 7 8 9 10 11 12
C
1 2 3 4 5 6 7 8 9 10 11 12
0 1 2 3 4 5 6 7 8 9 10 11 12
B

Figures 1A and 1B

Then the slanting line will cut the middle parallel at a point distant $C = (A + B) / 2$ from the square end. These facts are equally true for any trapezoid, such as the one shown by the dotted line. Figure 1B shows this dotted, slanting end cutting the upper base at $A = 7$, the lower base at $B = 5$, and the middle parallel at $C = (A + B) / 2$ or $(7 + 5) / 2 = 6$.

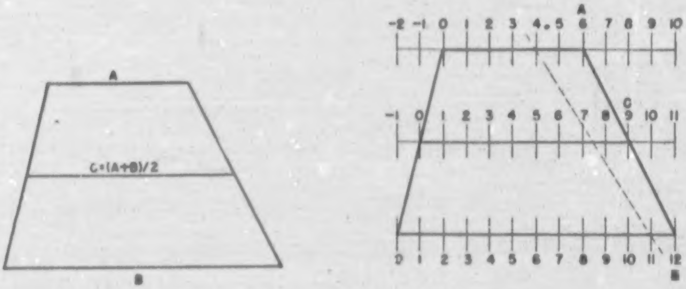
Figures 2A and 2B are identical with Figures 1A and 1B except that they have been turned vertically. Figure 2B also contains a third straight line and illustrates how this simple rudimentary alignment chart could be put to use by placing a straightedge or straight line through numbers on the outside scales which it is


B
C=(A+B)/2
A
12 11 10 9 A 8 7 6 5 4 3 2 1 0
12 11 C 10 9 8 7 6 5 4 3 2 1
12 B 11 10 9 8 7 6 5 4 3 2 1 0

Figures 2A and 2B

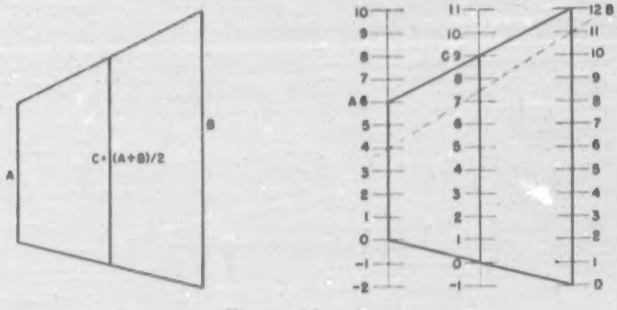
desired to add. One-half the sum of these numbers is then given by that number on the middle scale which is cut by the straightedge.

Figure 3A is a more general trapezoid with two slanting ends and with bases A and B and a middle parallel of length $C = (A + B) / 2$. Figure 3B shows three scales laid out on the parallels as they were in Figure 1B, with the zero points of the scales over at the left and lying on the oblique line which forms the left side of the trapezoid. As in the preceding case, the length of the middle parallel is $C = (A$ [illegible] $B) / 2$ and the middle scale will be cut at that number by the slanting end line.


A
C=(A+B)/2
B
A
-2 -1 0 1 2 3 4.5 6 7 8 9 10
C
-1 0 1 2 3 4 5 6 7 8 9 10 11
0 1 2 3 4 5 6 7 8 9 10 11 12
B

Figures 3A and 3B

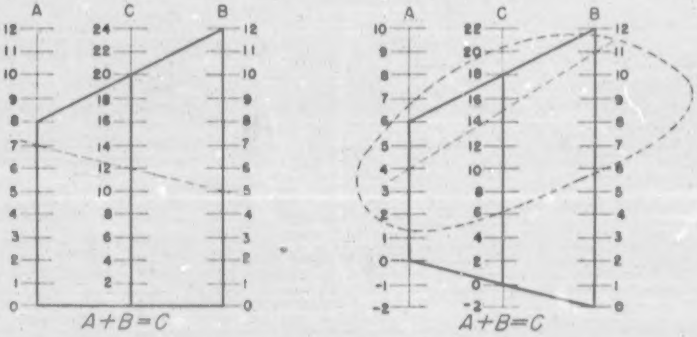
In Figures 4A and 4B the parallel sides have been made vertical; in Figure 4B a straight line has been drawn through the random points $A = 4$, $B = 11$ which crosses the middle parallel at the number $C = (A + B)/2 = (4 + 11)/2 = 7\frac{1}{2}$.


A
C=(A+B)/2
B
10 9 8 7 A6 5 4 3 2 1 0 -1 -2
11 10 C9 8 7 6 5 4 3 2 1 0 -1
12 B 11 10 9 8 7 6 5 4 3 2 1 0

Figures 4A and 4B

Four things should be acceptable to the reader by now.

(1) It should be clear that if the numbers marked on the middle scale are *doubled*, then a straight line through the A and B numbers he wishes to add cuts the middle scale at the true value of the sum, not at one-half that value. Figures 5A and 5B illustrate this point and are each complete alignment charts for the formula $A + B = C$ for their respective ranges of A, B, and C.



Figures 5A and 5B

(2) He should observe that the *zero* points of the three scales for this very simple chart are always collinear. This is proper because, when $A = 0$ and $B = 0$, $C = A + B = 0$ and the three values should lie on a straight line.

(3) He should not hesitate to extend any of the scales up or down through positive or negative values as far as need be to cover the range of variables which interests him. In so doing he will simply be creating a new trapezoid with the same properties that the first one had.

(4) He should realize that any restricted portion of the chart, such as that shown in Figure 5B, is, standing alone, a perfectly good and usable chart even though the zero points of the scales may happen to lie *outside it*.

ILLUSTRATIVE PROBLEM (FIG. 6)

We wish to have an alignment chart for the equation $A + B = C$ on which the A-scale runs from 5 to 10 and the B-scale from 25 to 30. How should the middle parallel scale be graduated?

Solution. The *A*-scale zero point is 5 units below the base line of the chart. The *B*-scale zero point is 25 units below the base line of the chart, or twenty units lower than the *A*-scale. The *A*, *B*, and *C* zero points lie on a straight line, as we observed before, because when $A = 0$ and $B = 0$, $C = A + B = 0$. Hence, the *C*-scale zero point is halfway between the *A*-scale and *B*-scale zero points or 15 units below the base line of the chart. Hence the base line of the chart will cut at number 30 on the *C*-scale. Do not forget that, to make the *C*-scale read the direct sum and not half the sum of $A + B$, the graduations should be twice as close as the *A* and *B* scale graduations.

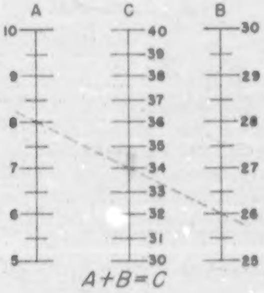


Figure 6

Alternative solution. The lowest number on the *C*-scale (on the base line) will be the sum of the lowest numbers on the *A*- and *B*-scales, or 30. Graduations are then made twice as close as those on the *A*- and *B*-scales, so the middle scale runs from 30 to 40.

ILLUSTRATIVE PROBLEM (FIG. 7)

We wish to have an alignment chart for the equation $A + B = C$, on which the *A*-scale runs from -10 to $+20$ and the *B*-scale runs from -5 to 35.

Solution. We can place the three zeros of the *A*-, *B*-, and *C*-scales on a straight line parallel to the bottom of the sheet and complete the chart as before, with the *C*-scale twice as densely graduated as the *A*- and *B*-scales, and show in the completed chart only such values of *A*, *B*, and *C* as were included in the given ranges. The diagram will then be badly lopsided, the *A*-scale hanging 5 units below the end of the *B*-scale and the *B*-scale rising 15 units above the end of the *A*-scale.

Preferable solution. Plan to have the three zeros on a slanting line. The *B*-scale is 10 units longer than the *A*-scale. Plan to have the *B*-scale project 5 units higher on each end than the *A*-scale. Then the lowest *B*-graduation, $B = -5$, appears 5 units below the lowest *A*-graduation, $A = -10$, and the *B* zero point will be 10 units below the *A* zero point. The *C*-scale zero point will be halfway between the *A* and *B* zeros on the middle scale and hence 5 units above the *B*-zero point. Make the remaining graduations on the *C*-scale twice as close as those on the *A*- and *B*-scales.

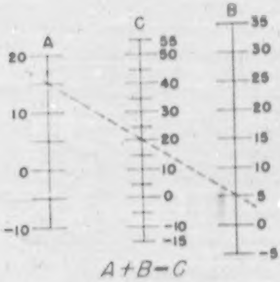


Figure 7

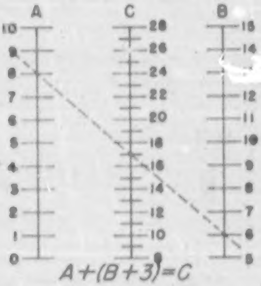


Figure 8

Illustrative Problem (Fig. 8)

We wish to construct an alignment chart for addition on which the *A*-scale runs from 0 to 10 and the *B*-scale runs from 5 to 15. *B* is also always to be increased by three before being added to *A*. That is, we wish to have an alignment chart for finding the value of *C* where $C = A + (B + 3)$.

Solution. Substitute $B' = B + 3$ where B' runs from 8 to 18, because *B* ran from 5 to 15. Solve the problem as has been done before for the relation $C = A + B'$. Then diminish each B' graduation by 3 to find the corresponding *B* graduation. Graduate only in *B*.

The limitations of the type of chart we have just been studying are apparent. An alignment chart is always used in connection with a formula which, directly or indirectly, has practical significance. Hence, the ranges of the *A*- and *B*-scales may differ very greatly. Such a diagram will be very lopsided if it is made according to the method used above. Hence the question arises as to whether the longer of these scales could not be squashed down or the shorter expanded or both steps taken until they were equal in length and filled perfectly the sides of a rectangular chart of specified dimensions. The graduations of a uniform scale after condensation or expansion would still be uniform among themselves throughout the length of that scale but closer or farther apart than the graduations of a different scale which had not been so treated. Thus, if it were desired to reduce a scale whose length was 40 inches into one whose length was 20 inches, the distance of each graduation from the zero point of its scale would have to be multiplied by ½. This relationship can be expressed in an equation.

Let $\overline{A_1A_2}$ be the *distance* between any two graduations A_1 and A_2.

Then

$$\overline{A_1A_2} = m_A\ (A_1 - A_2)$$

$$\overline{B_1B_2} = m_B\ (B_1 - B_2)$$

$$\overline{C_1C_2} = m_C\ (C_1 - C_2)$$

The first of these three equations simply says that the distance $\overline{A_1A_2}$ between graduations A_1 and A_2 is not given by $A_1 - A_2$ but by $m_A\ (A_1 - A_2)$. This is the same as saying that the distance between graduations A_1 and A_2 is greater or less than $A_1 - A_2$ according to whether m_A is greater or less than 1. The other equations make the corresponding statements about the other scales. m_A, m_B, m_C are each called the scale multipliers for their scales and each will be used to enlarge or reduce its scale until it just fills the space available for it.

The question which then arises is whether, after thus condensing one or both of the *A*- and *B*-scales, we can still use an arrangement of two parallel *A*- and *B*-scales with a middle parallel *C*-scale to give an alignment chart for the equation $A + B = C$, as we did above. The answer is that three parallel scales can be so used but the inner scale will not be centrally placed.

Let S be the distance between the A- and B-scales, or the available chart width, and let $h_{A,C}$ be the distance from the C-scale to the A-scale.

Then $$h_{A,C} = \frac{m_A \cdot S}{m_A + m_B}.$$

Thus we know where to place the C-scale once m_A and m_B are known.

ILLUSTRATIVE PROBLEM

We wish to construct a large alignment chart for the equation $A + B = C$ on a 19″ x 24″ sheet with about a two-inch margin. For various reasons we propose to condense the B-scale by a scale multiplier $m_B = \frac{1}{2}$, but to keep the A-scale its full size, so that $m_A = 1$. Where should the C-scale be placed?

Solution. $$m_A = 1,$$ $$m_B = \tfrac{1}{2}.$$

Since we wish about a two-inch margin, the distance between scales should be taken as 15. Then

$$S = 15.$$

Then we have $$h_{A,C} = \frac{S \cdot m_A}{m_A + m_B}$$ $$h_{A,C} = \frac{15 \cdot 1}{\frac{1}{2} + 1}$$ $$h_{A,C} = \frac{15 \cdot 2}{3} = 10.$$

The remaining question is whether the C-scale will be condensed by some multiplier m_C when m_A is different from m_B and if so what its multiplier m_C should be.

The formula for m_C can be worked out and will be found to be

$$m_C = \frac{m_A \cdot m_B}{m_A + m_B}.$$

ILLUSTRATIVE PROBLEM

We wish to construct a chart for the equation $A + B = C$ with the A- and B-scales 15 inches apart. Assume it is necessary to condense the *A*-scale by a multiplier 1/3 and to expand the *B*-scale by a multiplier 3. Where will the *C*-scale lie and what will be its multiplier?

Solution. $m_A = 1/3, m_B = 3, S = 15.$

Hence
$$h_{A,C} = \frac{S \cdot m_A}{m_A + m_B}$$
$$= \frac{15 \cdot 1/3}{1/3 + 3}$$
$$= \frac{15}{3} \cdot \frac{3}{10}$$
$$= \frac{3}{2}$$
$$m_C = \frac{m_A \cdot m_B}{m_A + m_B}$$
$$= \frac{1/3 \cdot 3}{1/3 + 3}$$
$$= \frac{1}{10/3}$$
$$= \frac{3}{10}$$

ILLUSTRATIVE PROBLEM (FIG. 9)

We wish to construct an alignment chart to represent the formula $A + B = C$, where *A* is to range from 0 to 100 and *B* is to range from —20 to 60. The largest piece of paper available is 19″ x 24″ and a two-inch margin is desired.

Solution. Let us place the *A*- and *B*-scales 15 inches apart, and let each scale be 20 inches high. The *A*-scale will have a multiplier 1/5. The *B*-scale will have a multiplier 1/4.

Then
$$h_{A,C} = \frac{S \cdot m_A}{m_A + m_B}$$
$$= \frac{15 \cdot 1/5}{1/5 + 1/4}$$
$$= 6\tfrac{2}{3}$$

$$m_C = \frac{m_A \cdot m_B}{m_A + m_B}$$

$$= \frac{1/5 \cdot 1/4}{1/5 + 1/4}$$

$$= 1/9.$$

The lower edge of the chart runs through $A = 0$, $B = -20$, and hence through $C = -20$.

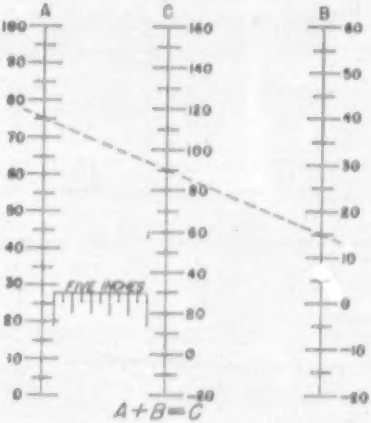

A
C
B
100
90
80
70
60
50
40
30
20
10
0
160
140
120
100
80
60
40
20
0
-20
60
50
40
30
20
10
0
-10
-20
FIVE INCHES
A+B=C

Figure 9

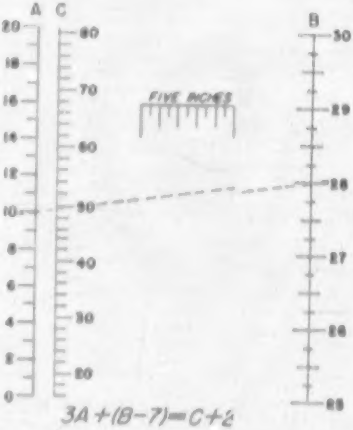

A
C
B
20
18
16
14
12
10
8
6
4
2
0
80
70
60
50
40
30
20
30
29
28
27
26
25
FIVE INCHES
3A+(B−7)=C+2

Figure 10

ILLUSTRATIVE PROBLEM (FIG. 10)

We wish to make an alignment chart which will fill a rectangle about 15″ x 20″ to represent the equation $3A + (B - 7) = (C + 2)$. A ranges from 0 to 20; B from 25 to 30.

Solution. The given equation can be reduced to the form

$$3A + B = C + 9.$$

We can make a substitution.

Let $A' = 3A$

$B' = B$

$C' = C + 9.$

Our given equation can then be written

$$A' + B' = C',$$

where A' ranges from 0 to 60; B' ranges from 25 to 30. We are well equipped to make a chart for this equation in A' and B'.

Clearly
$$\begin{aligned} m_{A}' &= 1/3 \\ m_{B}' &= 4 \\ m_{C}' &= \frac{4 \cdot 1/3}{4 + 1/3} \\ &= 4/13 \\ h_{A',C'} &= \frac{15 \cdot 1/3}{4 + 1/3} \\ &= 15/13. \end{aligned}$$

Thus we have found where the C'- (and hence C-) scale will lie, and have found the C' multiplier m_{C}'. We could easily graduate the scales in values of A', B', and C', but hope to be able to graduate in A, B, and C, our original variables. This is done as follows:

Let $$A = 0.$$

Then $$A' = 3A = 0.$$

Since $A = 0$ occurs at the base line of the chart, that is where $A' = 0$ occurs.

Let $$A = 5.$$

Then $$A' = 15.$$

$A' = 0$ occurs at the base line, and $A' = 15$ will occur above it at a distance $(1/3)\ 15 = 5$; so that is where the graduation $A = 5$ lies. All other A graduations can be similarly worked out. It will be found that the twenty units of A turn out to be evenly distributed throughout the twenty units of length of the A-scale, just as desired. Nevertheless, *the steps taken through A' and the evaluation of m_{A}' are unavoidable.*

The B'-scale is identical with the B-scale and offers little trouble.

The C-scale is found as follows: The lowest value of C is given by adding the lowest values of A and B, and equals 16. Hence the lowest value of C' is 25.

Let us now find, for example, where $C = 20$ occurs.

$$C' = C + 9$$
$$C' = 29.$$

The distance of $C' = 29$ above $C' = 25$ (at base line) is given by

$$m_{C'}(29 - 25) = \frac{\text{[illegible]}}{13}(29 - 25)$$
$$= \frac{16}{13}$$
$$= 1.23.$$

Although we have now discussed the general mechanics of construction and operation of the three-parallel-lines type of chart, we have not yet touched that particular application which makes it so very valuable. The chart would scarcely be worth the attention we have given it if it could not be used for multiplication as well as for addition.

ILLUSTRATIVE PROBLEM (FIG. 11)

An alignment chart to fill a 10″ x 15″ rectangle is desired for the equation $U \cdot V = W$, where U runs from 1 to 1000 and V runs from 1 to 1000.

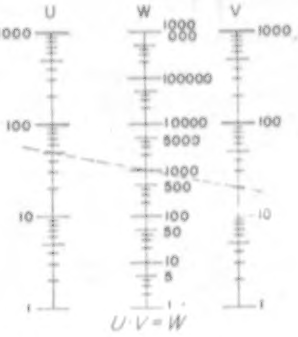

U
W
V
1000
1000 000
1000
100000
100
10000
5000
100
1000
500
10
100
50
10
10
5
1
1
1
U·V = W

Figure 11

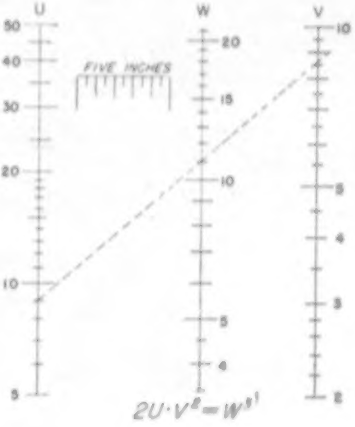

U
W
V
50
40
30
20
10
5
FIVE INCHES
20
15
10
5
4
10
5
4
3
2
2U·V² = W³

Figure 12

Solution. Taking the logarithm of both sides of the equation we have

$$\log (U \cdot V) = \log W$$

$$\log U + \log V = \log W.$$

Let

$$A = \log U$$

$$B = \log V$$

$$C = \log W,$$

where A and B both range from log 1 to log 1000 or from 0 to 3.

Then, on substitution, our given equation becomes one with which we are familiar; namely

$$A + B = C.$$

Then

$$m_A = 6.66$$

$$m_B = 6.66$$

$$m_C = 3.33$$

$$h_{A,C} = \frac{m_A.S}{m_A + m_B}$$

$$= \frac{(6.66)\cdot 15}{6.66 + 6.66}$$

$$= \frac{15}{2}$$

$$= 7.5,$$

and the C-scale will lie in the center of the chart. We wish to graduate eventually in the original variables U, V, W. This can be done through the equations

$$A = \log U$$

$$B = \log V$$

$$C = \log W.$$

Thus the graduation $U = 30$ implies an A graduation:

$$A = \log 30$$

$$= 1.4771.$$

The lowest A reading on the base line is

$$\begin{aligned} A &= \log 1 \\ &= 0. \end{aligned}$$

The difference between the readings $A = \log 30$ and $A = \log 1$ is 1.48.

But $m_A = 6.66$, so that the *distance* between the readings is

$$6.66 \times 1.48 = 9.85.$$

Thus the graduation $U = 30$ occurs 9.85 above the base line.

ILLUSTRATIVE PROBLEM (FIG. 12)

We wish to construct an alignment chart on a 15″ x 20″ rectangle for the formula $2U \cdot V^2 = W^3$, where U ranges from 5 to 50 and V from 2 to 10.

Solution. Taking the logarithm of both sides we have

$$\log [(2U) \cdot (V^2)] = \log (W^3)$$

$$\log (2U) + \log (V^2) = \log (W^3)$$

$$\log 2 + \log U + 2 \log V = 3 \log W.$$

Let
$$\log 2 + \log U = A$$

$$2 \log V = B$$

$$3 \log W = C.$$

Our original equation becomes one with which we are familiar; namely,

$$A + B = C,$$

where A ranges from $(\log 2 + \log 5)$ to $(\log 2 + \log 50)$. Hence the total range in A is given by $(.30 + 1.70) - (.30 + .70) = 1$. B ranges from $2 \log 2$ to $2 \log 10$, giving a total range of 1.40.

Then $m_A = 20, m_B = 14.28.$

Hence

$$\begin{aligned} m_C &= \frac{20(14.88)}{20 + 14.88} \\ &= 8.33. \end{aligned}$$

The position of the C-scale is given by

$$h_{A,C} = \frac{15(20)}{34.38} = 8.75.$$

How should the graduation $W = 5$, for instance, be placed?

We have

$$\begin{aligned} C &= 3 \log W \\ &= 3 \log 5 \\ &= 3(.70) \\ &= 2.09. \end{aligned}$$

The lowest value of W (on the base line) corresponds to the lowest values of U and V (on the base line); hence, for W on the base line we have

$$\begin{aligned} 2 \cdot 5 \cdot 2^2 &= W^3 \\ 3.40 &= W \end{aligned}$$

but
$$\begin{aligned} C &= 3 \log W \\ &= 1.60. \end{aligned}$$

The difference between these two values of C is 0.49.

Then the distance between the two readings, which is also the distance from the base line to the graduation $W = 5$, is given by

$$\begin{aligned} m_C(.49) &= 8.33\,(.49) \\ &= 4.12. \end{aligned}$$

Thus the graduation $W = 5$ occurs 4.12 units above the base line.

The illustrative problems we have solved above all have about the same approach. The rules noted below should be followed carefully.

(1) The equation for which the chart is to be made *must be put* in the *simple form* $A + B = C$, by the use of the simple variables A, B, C. This may require the application of logarithms and several substitutions.

(2) The *ranges* for the simple variables A, B, C are computed from the ranges of the original variables in conjunction with the equations of substitution between them and the new, simple variables.

(3) The *multipliers* m_A, m_B for the simple variables A and B can now be determined since the height of the chart is known and the ranges of the simple variables have just been found in (2).

(4) The *position* of the inner scale can now be found from the relation

$$h_{A,C} = \frac{m_A \cdot S}{m_A + m_B},$$

where S is the prescribed distance between the outer scales.

(5) The *multiplier* m_C can be evaluated from the relation

$$m_C = \frac{m_A \cdot m_B}{m_A + m_B}.$$

(6) We now calibrate the scales in terms of the original variables by finding the values of the simple variables to which they correspond and learning where these latter values would lie. Naturally we choose those convenient values of the original variables which we wish to have appear on the chart.

ILLUSTRATIVE PROBLEM (FIG. 13)

We wish to construct an alignment chart on a 19″ x 24″ sheet with about a 2-inch border for the equation

$$(U + 2)^{1/3} \cdot V^{.78} = W + 4.$$

The range of U is from 0 to 28. The range of V is from 1 to 30.

Solution. Taking logarithms of both sides we have

(1) $1/3 \log (U + 2) + .78 \log V = \log (W + 4)$

Let

$$A = 1/3 \log (U + 2)$$

$$B = .78 \log V$$

$$C = \log (W + 4).$$

Then we have achieved the *simple form:*

$$A + B = C.$$

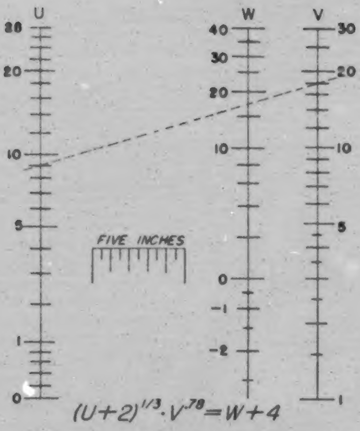

U
W
V
28
20
10
5
1
0
40
30
20
10
0
−1
−2
30
20
10
5
1
FIVE INCHES
$(U+2)^{1/3} \cdot V^{.78} = W+4$

Figure 13

(2) The range of the *simple variable A* is from 1/3 log (2) up to 1/3 log (30), or from .10 to .49. The range of the simple variable *B* is from .78 log (1) up to .78 log (30) or from 0 to 1.15. The total range in *A* is .39. The total range in *B* is 1.15.

(3) Then $m_A = \dfrac{20}{.39} = 51.02$

$$m_B = \frac{20}{1.15} = 17.36.$$

(4) The position of the *C*-scale is given by

$$h_{A,C} = \frac{S \cdot m_A}{m_A + m_B} = \frac{15 \quad (51.02)}{51.02 + 17.36}$$
$$= 11.19.$$

(5) The multiplier m_C is given by

$$m_C = \frac{m_A \cdot m_B}{m_A + m_B}$$
$$= \frac{(51.02)(17.36)}{(51.02 + 17.36)}$$
$$= 12.95.$$

Let us find, for example, where the graduation $U = 1$ would lie.

When $U = 1$, $A = 1/3 \log (1 + 2) = .16$. We saw that the lowest A reading is .10, and the difference between this reading and the lowest A reading is .06. The *distance* between these readings (or the height above the base line of the A graduation corresponding to $U = 1$) will equal the product of the multiplier m_A and the difference of these readings, or $U = 1$ occurs at a height of $(51.02)(.06) = 3.06$ above the base line. Other U-graduations are computed in identical manner.

It is important to systematize computations. Once a routine has been set up the labor for a difficult chart will not be as great as it at first seems. A measuring-scale graduated in decimals is helpful but not indispensable for laying off the results.

It should be emphasized that the material in this article is not intended to be a basis for teaching the subject of alignment charts to the average high school student. It should, however, prove an adequate guide to those able, ambitious, and aggressive young people who enjoy wrestling with new problems once they know that a fair fight promises a liberal reward.

THE CONSTRUCTION AND USE OF HOMEMADE INSTRUMENTS IN INDIRECT MEASUREMENT

Virgil S. Mallory

It is probably true that teachers of mathematics in general and those in the senior high school in particular are more inclined to emphasize abstract theory and manipulations than the practical uses of mathematics. There is considerable truth in the statement that most teachers of mathematics consider that the only equipment necessary in the classroom is a blackboard, a piece of chalk, and a bit of string with which to draw circles. That this attitude on the part of mathematics teachers is detrimental to efficient teaching should be obvious to any intelligent person.

Knowledge of the transfer of training and how transfer takes place through the conscious effort of the teacher to initiate real life situations, or situations so closely related to those in real life that the student cannot fail to see the application, must cause many *real* mathematics teachers to realize the necessity of using many practical applications of mathematics. The oft-repeated and justifiable criticism that a student frequently cannot apply his mathematics in a field as closely related to it as physics emphasizes this need. Yet frequently one must go to the science laboratory in order to find such obvious applications of mathematics as are afforded by studying the relative volumes of the cylinder, cone, and sphere through actual measurement instead of only through theoretically derived formulas, a comparison which should be made in the mathematics classroom. Moreover, it is not unusual to see a teacher present *board measurement* (where it is taught) without a single piece of board in the classroom with which to show a *board foot* and without any attempt to use the doors and desk tops in the classroom actually to measure and examine board feet.

The reason for this apathy is not difficult to find. Many teachers of mathematics have not had any engineering or shop training and do not know the applications of mathematics that should unite theoretical and practical work. On the other hand, many who do know about the possible applications fail to realize that for the students book problems and book explanations cannot take the place of laboratory demonstrations and field work. This is a fact that science teachers understand full well. What is more, laboratory and field work, with the necessary planning and accumulating of material, is more difficult and time consuming than assigning the "next ten problems in the book." Only a real teacher is willing to expend the extra time and effort required.

Unless the teacher has definitely in mind the objectives he hopes to attain in presenting field problems or laboratory work in mathematics, the work will become aimless and lose much of its educative worth. Although no attempt will be made here to catalogue all of them, there are definite values which this type of work can contribute to the education of the child, and which can be readily attained. Some of these are:

(1) An appreciation of the reality of mathematics and of the way it functions in everyday life.

(2) An appreciation of the approximate nature of all measurement and of the need for the intelligent determination of the number of significant figures to be retained in a final result.

(3) An appreciation of the omnipresence of the opportunity for error, its possibility not only in calculation but in application, and of ways to discover and avoid such errors.

Thus the child soon learns to look for error in the calibration of protractors and yardsticks, in the use of flexible tapes, in frequent applications of the unit (such as the use of a footrule vs. a 100-foot tape in measuring a length), in careless reading of measures, in scale drawing through use of a soft pencil, and in many other ways.

(4) An appreciation of the function relation that is direct and dramatic.

Thus the child who discovers that an error of less than a degree in laying out an angle on his scale drawing may cause a large variation in the apparent height of the tree or flagpole has a real appreciation of the fact that *in a triangle the side opposite varies with the angle.*

(5) An acquaintance with and skill in using various simple measuring instruments.

(6) A liking for mathematics and a desire to learn more about a subject that has such interesting applications.

When this type of work is properly conducted, children generally enjoy it. Thus it provides an excellent device for motivating the more theoretical or manipulative work.

Field work in measurement can be taught in every grade from seven to twelve. Indirect measurement with solutions by scale drawings provides a natural motivation for the general mathematics of grades seven to ten; in later grades the trigonometric functions can be used to solve problems, although the value of scale drawings in the actual solutions should not be minimized. No class in trigonometry is complete without field work.

Entirely aside from the cost involved in providing expensive equipment, there are both practical and pedagogical reasons why inexpensive transits made in the school shop or at home are better for classroom use than are the ready-made ones. The most important reason is that the protractors, the working part of the measuring device, are seen by the child, and are not complicated by adjusting devices, a knowledge of which he must master. Furthermore, costing less than seventy-five cents each and readily made in the school shop or at home, the instruments can be provided in quantities sufficient for class use.

The class that looks at and admires the store-bought transit while the teacher demonstrates it never attains that facility and appreciation of measurement which can be obtained by having every four members of a class work as a crew with an instrument itself. The use of cloth tapes will eliminate the expense of the repair of tapes and instruments which is inevitable with more expensive instruments. By working in crews of four, students develop an esprit de corps and a healthy sense of competition that have most beneficial results.

Teaching devices for any type of school, while informal, require the maintenance of ordered discipline. Class instruction covering every detail of the work, the assignment of pupils to crews, the measurements to be made, the scale to be used, and what each crew is to do when its measurements are completed should be given before the class leaves the classroom. The measurements to be made should be confined to a limited area so that the teacher can easily oversee the groups and give the necessary individual instruction. There will be more of this instruction and correction of procedures for separate crews the first time the class goes out than later on.

As far as possible each crew should be composed entirely of boys or entirely of girls. The teacher should make certain that everyone has a chance, at some time or another, to experience all the tasks from sighting the object to using the tape. Each member of a crew should have a notebook in which he makes a freehand drawing of the problem before leaving the classroom and on which he places the measurements as he obtains them.

To be most successful the field work should be preceded by lessons in scale drawing so that the students will acquire the ability to transfer their notes to a carefully made scale drawing. Practice should also be given in changing measurements to the scale agreed on. Thus if the scale is 1 : 20, how long a line on the paper will be represented by a measurement of 93′ 6″?

In making the scale drawings the most satisfactory method is for each student to be provided with a sharp, hard pencil, a compass, a ruler, a protractor, and an inexpensive drawing set. The drawing set may consist of a drawing board (about 14″ x 18″ or larger), a T-square, a 45°-right triangle, and a 60°-right triangle. Such equipment, made of wood, may be bought at little cost from school supply houses or may be made of plywood in the school workshop.

Each crew should have a homemade school transit provided with both vertical and horizontal protractors, a 25-, 50-, or 100-foot cloth tape (sometimes obtainable in 5 and 10 cent stores), and half a dozen 1/8″ x 6″ dowel sticks (obtained in a hardware store) for the head tape-man to mark the point where the end of the tape comes and for the rear tape-man to pick up. The number of these dowels that the rear tape-man has at the end of a measurement will tell the number of times the full tape measurement is contained in the distance measured.

Even if drawing board, T-square, and triangles cannot be obtained, the project need not be given up. The next most satisfactory method is to use large sheets of graph paper. The grid lines will furnish perpendicular lines and measures to aid in the construction. Graph paper ruled ten lines to the inch is also convenient to use in making measurements to tenths of an inch and estimates to hundredths of an inch, always a desideratum in calculated results.

THE HOMEMADE TRANSIT

Types of problems. The following five problems[1] show the types that can be solved with the homemade transit. In each case the student should solve, by scale drawing, a similar problem so that he may know how to proceed with his solution as soon as he has obtained the field measurements. If the class has studied the trigonometric functions, that method may be used as a check on the drawing. The teacher should know the answers to the field problems but obviously should not indicate them to any crew until all have completed their work.

1. Find the height of the school flagpole (or a tree or the school building).

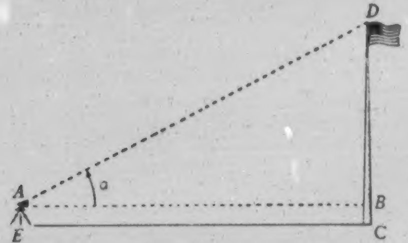


Figure 1

The problem involves measuring, with the vertical transit (Fig. 1), angle *a*, and with the tape the distance *AB*, and the height *CB* of the transit, and is solved by making a scale drawing of a right triangle. In measuring *AB* it should be noticed that the line *AB* is horizontal and does not follow the contour of the ground. In measuring the height of the transit, notice that this is not taken at *AE,* but that a pupil determines the point *B* at which the horizontal line *AB* intersects the pole. The flagpole or tree should, in this problem, be on horizontal ground and perpendicular to the ground. Similar problems using the right triangle in either the vertical or the horizontal plane can be easily devised.

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[1] Problems and drawings from Mallory, *Mathematics for Everyday Affairs*, used by permission of the publisher, Benj. H. Sanborn and Co.

2. A surveyor wished to find all the dimensions of a triangular field which included part of a swamp. He also wished to know the perpendicular distance from *A* to *BC* (*AD*). He can measure angles *B* and *C* with his protractor and *BC* with his tape. (See Fig. 2.)

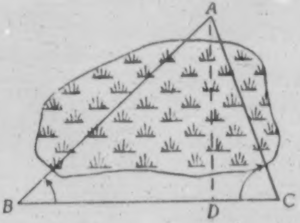

Figure 2

This problem involves the solution of an oblique triangle, given two angles and the included side. The horizontal protractor on the transit is used in measuring the angles. A scale drawing solves the problem.

Natural terrain for the problem would, of course, be a swamp, a pond, or a woods with point *A* marked by a tall tree. Lacking these, an imaginary swamp or pond can be used or a point *A* so far away that its distance is more easily found by measuring the parts indicated above than by a direct measurement.

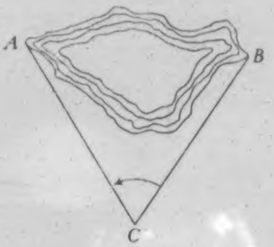

Figure 3

3. Two boys swam from a point on one shore of a lake to a point on the opposite shore, and they wished to know how far they had swum. They found it by measuring *AC* and *BC* and angle *C*.

This problem involves the solution of an oblique triangle, given two sides and the included angle. The horizontal protractor on the transit is used and a scale drawing solves the problem.

Instead of measuring the distance *AB* across a swamp or lake (Fig. 3), the distance *AB* through some school building may be measured. The distance through a small hill or woods can also be used instead to show how such distances may be measured.

4. Some Boy Scouts had a camp (*C*) in the woods. They knew that town *A* was directly west of camp and 3 miles from camp, that town *B* was 4¼ miles from camp, and that the distance between *A* and *B* was 3¾ miles. They wished to find the compass bearing in order to go directly from *C* to *B*, knowing that the direction was something east or west of north. (See Fig. 4.)

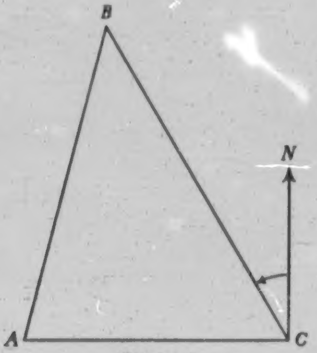

Figure 4

This problem involves the measurement of three sides of an oblique triangle. Practical problems of this kind are not as easily found as in the other cases. The solution is, of course, by scale drawing.

5. A surveyor wished to find the height above ground of a church spire. He found it by measuring side *AB* and angles *A* and *DBC*. (See Fig. 5.)

This problem involves an oblique triangle with two angles and the included side to be measured and the solution by means of a scale drawing. Any situation where a building has a tower or a cupola set back from the edge of the roof is suitable. There are

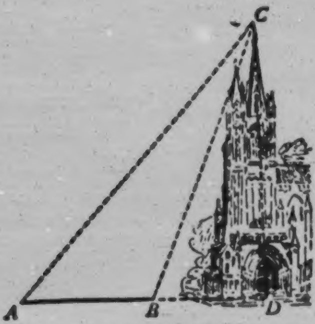

Figure 5

many modifications of this problem: (a) At the seashore or at a lake (or on a level plane) to measure the height of a cliff or tower from which observations are made by using the angle of depression of two buoys or other objects in the line of sight and a known distance apart. (b) In (a), to find the distance apart of the buoys or objects when the height of the cliff or tower is known. (c) To find the height of a flagpole on a hill by measuring *AB* and the angles of elevation of its top from points *A* and *B* and of its foot from either point *A* or *B*.

HOW TO MAKE A SIMPLE TRANSIT[2]

The table top. Figure 6 shows the table top of the transit. It is a piece of wood (three-ply wood is best) 12¼ inches square and ⅜ inch thick. Draw a circle with 5-inch radius on this. Drill a ¾-inch hole through the center and drill and countersink six $\frac{3}{16}$-inch holes spaced as shown at *C* in the drawing.

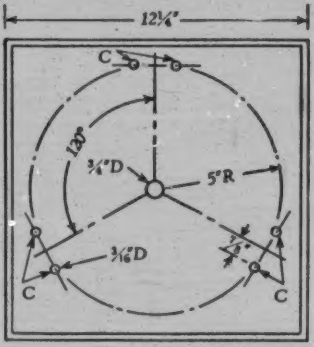

Figure 6

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[2] Taken from Mallory, V. S., *Mathematics for Everyday Affairs*, by permission of the publisher, Benj. H. Sanborn and Co.

The legs for the transit are three dowel sticks, 3/4 inch in diameter and 3 feet long. They can be obtained at any hardware store. Figure 7 shows how one end of the dowel stick is shaped to be attached to the table top. Figure 8 shows how the legs are attached to the table top by means of 1-inch by 1-inch angle irons. Use 3/4-inch by 3/16-inch flat-head stove bolts to fasten the legs to the angle irons and table top.

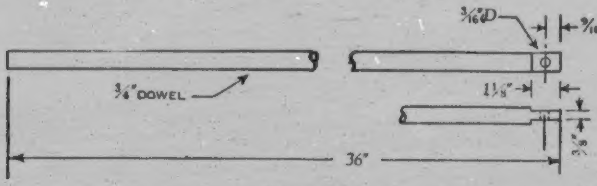

Figure 7

The horizontal protractor. Glue a paper protractor (Fig. 9) to a piece of heavy cardboard or pressed wood 12-inches square. You can make the protractor from polar coordinate paper. Mark it from 0° to 180° in each direction. Drill a hole with 3/4-inch diameter in the center as shown.

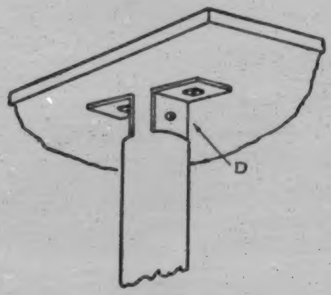

Figure 8

The pointer is a 1/4-inch dowel stick (Fig. 10) with one end sharpened as shown. This fits in the 1/4-inch hole drilled in the 3/4-inch dowel stick shown in Figure 11. At *A* a screw eye is placed to attach a lead sinker for a plumb bob. The completed transit for measuring horizontal angles is shown in Figure 12.

The vertical protractor. To measure the angle of depression or the angle of elevation in finding the height of an object a vertical pro-

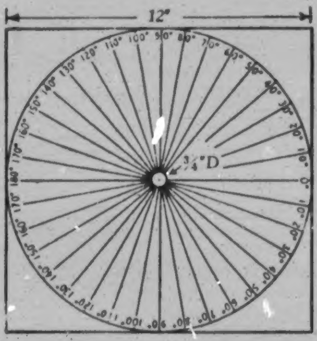

Figure 9

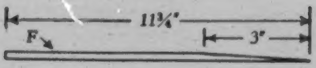

Figure 10

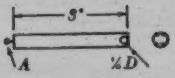

Figure 11

tractor, shown in Figure 13, replaces the horizontal protractor and pointer. A semicircular protractor marked from 0° to 90° in both directions is glued to a semicircular piece of stiff cardboard or pressed wood and a 3/16-inch hole (*K*) is drilled as shown in Figure 14. At *B* are screw eyes to be used for sighting.

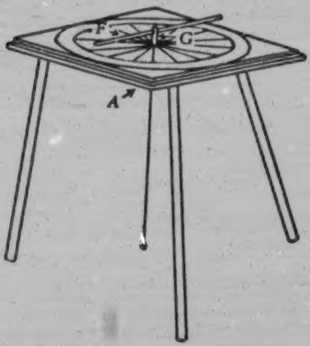

Figure 12

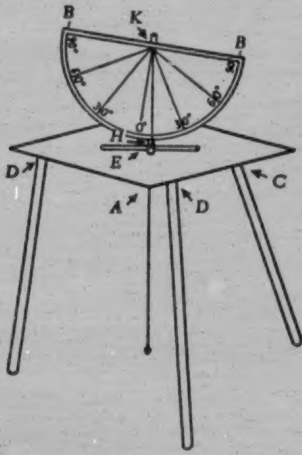

Figure 13

This protractor is fastened at *K* to the 3/4-inch dowel stick shaped and drilled as shown in Figure 15. The drawing in Figure 16 shows the details of fastening the protractor to the dowel stick. A sinker hung by a string at *K* (Figure 18) acts as a plumb bob to enable one to read the angle of elevation. Figure 17 shows a 1/4-inch dowel stick used to support the upright. (See E, Figure 13.)

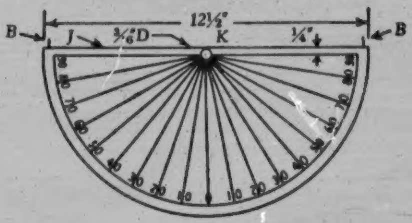

Figure 14

OTHER DEVICES FOR INDIRECT MEASUREMENT

Other homemade devices for making indirect measurement are perhaps better known.

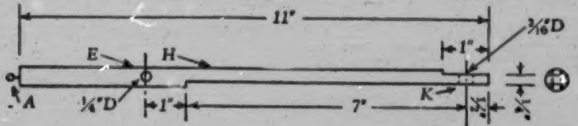

Figure 15

An isosceles right triangle of cardboard held so that one leg is horizontal and so that the hypotenuse can be used as a sighting edge will give the height of a tree when the observer stands at a point where the line of sight intersects the tree top.

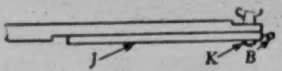

Figure 16

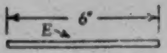

Figure 17

The measurement of the shadow of a yardstick and of the shadow of a tree at the same time will give the data necessary to determine the height of the tree.

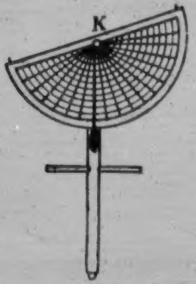

Figure 18

The use of the geometric square, cross-staff, drumheads, a wooden right triangle, and other historical devices,[3] not as obvious in their applications, can also be used for indirect measurements, as can the hypsometer and clinometer.[4]

The plane table,[5] merely a drawing board mounted on a tripod, and with some kind of sighting arrangement (an alidade; it may be only a ruler with two pins mounted for sighting), is an excellent device for making maps and determining inaccessible distances and angles. It is frequently used by the United States Army.

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[3] See Mallory, V. S. *New Plane Geometry* (*New Edition*), pp. 330-336. Benj. H. Sanborn and Co., Chicago, 1943.

[4] See Schuster, C. N. and Bedford, F. L. *Field Work in Mathematics.* American Book Co., New York, 1935.

[5] *Ibid.*, pp. 59-64; or Mallory, V. S., *New Plane Geometry* (*New Edition*), pp. 372-373. Benj. H. Sanborn and Co., Chicago, 1943.

This type of work is probably more difficult to arrange in the larger city high schools than in suburban or country schools where outdoor facilities are readily obtained. But even in a city high school the ingenious teacher, through the use of the classroom, corridor, auditorium, playground, or park, can accomplish a great deal.

A CLASSIFICATION OF MATHEMATICAL INSTRUMENTS AND SOURCES OF THEIR PICTURES

Henry W. Syer

TEACHERS who wish to enliven their mathematics instruction and make it more practical will welcome mathematical instruments which can be used for this purpose. However, there are so many hundreds of different machines that it is impossible to collect them all. Moreover, many of the interesting historical models are available only in large cities or at great expense. Therefore, the following list of illustrations of mathematical machines has been compiled.

Pictures should not be considered a substitute for the machines themselves, which should be brought into the classroom and given to the students to manipulate and examine at first-hand and as often as possible. Pictures should be considered a method to increase the total number of instruments with which the class can become familiar. They can be used in everyday instruction as the sources of problems in geometry and algebra, as correlative material for outside projects, for club programs, and for bulletin board displays.

The final classification of mathematical instruments given at the end of this article is complete and elastic enough to include any future instruments which may creep into my file. The sources of pictures listed, however, include only pictures large enough to be of use in the classroom.

PLANS FOR FUTURE DEVELOPMENT

There are two ways in which these pictures of instruments can be made available to high schools inexpensively. First, photographic, planographic, and photostatic reproductions could easily be made large enough for schools to mount on bristol board and keep in the art collection or mathematics office to be circulated to the teachers. The second method is even less expensive and calls for photographing the pictures in convenient sets on film strips which can be projected. One set of these film strips has already been completed and is now on sale.*

The first of these methods has the advantage that the pictures can be arranged in any order, can be used by many students or classes at the same time, and can be left on permanent display. The second method, however, has the advantage of very inexpensive duplication, easy control by the teachers, and the possibility of being seen by a large group at one time.

Concise, accurate descriptions of the history and use of the machines, together with their methods of operation, should accompany either the set of photographs or the film strip. Care must be taken that, in an attempt to be complete, one does not make these descriptions abstruse. They should be simply worded, and should give only essential information for quick understanding by secondary school students.

LISTS OF SOURCES

There are two chief sources of pictures of mathematical instruments: first, museums, commercial instrument companies, and private collectors; second, books on the history of mathematics in general or on calculating machines in particular. These will be found in the bibliography that follows and in the list of abbreviations under the classification of instruments.

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Classification of Mathematical Instruments, with Sources[*]

In the following classification will be found the name of the instrument, followed by a few selected sources of pictures large enough to be of use in the classroom.

Section I. Measurement Instruments

Part A. Laboratory Instruments

1. Discrete Units
   Hand Counter—Cenco Cat.
   Hand Multi-Counter—(67:8.2, Mul. Counter 2)
   Photo-Electric Counter
   Test Scorer—International Cat.
2. Length
   Gambey Comparator—(2:245)
   Electrolimit Comparator—P & W.
   Calipers—Cenco Cat.; B. & S. Cat.

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[*] Abbreviations used:
(2:245 = Book 2 of Bibliography, page 245.
(67:8.2 Mul. Counter, p. 2) = Book 67, section 8.2, part entitled "Mul. Counter," page 2.
B. & S. = Browne and Sharpe Co., Providence, R. I.
Cenco Cat. = Catalog of the Central Scientific Co., Chicago, Ill.
Comp. = Used in "From Comp" to mean picture supplied by the manufacturer.
International Cat. = Catalog of the International Business Machine Co.
K & E = Keuffel and Esser Co., New York City.
K W = Catalog, Kelvin White Co., Boston, Mass.
M.I.T. = Massachusetts Institute of Technology, Cambridge, Mass.
P & W = Pratt and Whitney, Toolmakers, Hartford, Conn.
Toledo = Toledo Scales Co., Toledo, Ohio.

   - Micrometer Screw–Cenco Cat.; B. & S. Cat.
   - Spherometer–Cenco Cat.
3. Mass and Weight
   - Historic Sets of Weights
      - Money Changers Weights–(52:123)
      - Pile de Charlemagne–(2:233)
   - Analytic Balances
      - Balance de precision–(2:248)
      - Modern Analytic Balances–Cenco Cat.
   - Commercial Scales–Toledo
      - Retail Scales, Metal Chart, Postal Scales, Brine Solution Scales, Platform Scales, Oil Barreling Scales, Counting Scales, Paper Ream and Box-board Scale, Yardage Scale, Motor Truck Scale, Double Pendulum Principle
4. Time
   - Clepsydra–(45:49)
   - Lamp Timekeeper–(45:54)
   - Sun Dials
      - Ancient–(66); Pillar Dial–(66); Modern– Cenco Cat.; On Gema Building, Graz; English; Cluster of Sun Dials at Bedales, Sussex, England; Pocket Sun Dial–(3); Brechte's Pendant Sun Dial (26:136)
   - Calendar, Coffin Lid of Tefabi–*Isis*, Vol. XVII.
   - Business Time Printers–International Cat.–(67:8.1, Int. P.S. 1)–Clock Systems, International Cat.
   - Stop Watches–Cenco and International Cat.
5. Speed and Velocity
   - Speed Indicator–Cenco Cat.

Part B. Applied Instruments

1. Surveying
   - Groma–So. Kensington Museum (52.236), (54:124)
   - Baculum–(55:347), (52:237)
   - Range Finder–(55:363), (2:71)
   - Semi-circle–(35:205)
   - Quadrant–(52:234, 240), (55:352)
   - Plane Table–(52:249), (2:57), K & E Cat., (34:208, 215)
   - Circumferentor–(52:248)
   - Theodolite–So. Kensington Museum–(2:52), (2:54)
   - Transit–(2:53), K & E Cat.
   - Level–(52:230), (3), (2:65), K & E Cat.
   - Leveling Rod–K & E Cat.
   - Surveying Compass–K & E Cat.
2. Navigating
   - The "Bowe"–(4:86)
   - Sextant–(2:99), K & E Cat.
   - Ship Log–K W Cat.

Nautical Compass—K W Cat.
Sounding Machine—K W Cat.
Artificial Horizon—K W Cat.
Course Protractor
Pelorus—K W Cat.
Current Meter—K W Cat.
Anemometer—K W Cat.

3. Astronomy
    Celestial Globe—(52:17), (55:365), (2:77)
    Nocturnal—(26:276)
    Volvelles (or Aequatoria)—(26:242)
    Orrery—(26:258)
    Astrolabe—Chaucer's Time, (55:348), Italian 1558—Front (23); Back; Front, (52:242); Back, (52:243); Front, (55:349); Astrolabe de G. Arsenius (2:80); Use in Mensuration (55:349); Use of (60)
    Armillae—(26:148)

Section II. Calculation Instruments

Part A. Graphical Methods

1. Drawing Instruments
    Proportional Dividers—(8), K & E Cat.
    Sector Compasses—(52:338); (52:342); (24:frontis.); (35:207); (35:208)
    Parallel Rules—(8), K W Cat.
    Pantograph—(8), K & E Cat.
    Equal Division Instrument—(19:18)
    Conographs—(19); Ellipsograph (2:37); Parabolagraph—(31:269); Circle Divisor (2:39)
    Co-ordinator—(31:269)
    Perspective Instrument—(19:239)
    Trace-Computer—(19:164)
2. Nomographs
    Vernier Scale
    Plain Scale—(4:159); Gunter's Scale (8)
    Oughtred's Circles of Proportion—(52:346)
    Alignment Charts—(10, 14, 17, 18, 40, 44)
    Network Charts—(same as Alignment Charts)
3. Lengths of Curves
    Curve Measurer—(22:65)
4. Areas
    Planimeter—Disc P—K & E Cat.; Corodi Precision Disc Planimeter—(6:70); Polar Planimeter—K & E Cat.; Amsler Polar Planimeter—(6:64); Boy's Polar Planimeter—(6:66); Rolling Planimeter—K & E Cat.; Planimeter (46:72); (31:200, 203); Radial Planimeter, K & E Cat.
    Integraph—Amsler Integraph (6:65); Amsler's Integraph—K & E Cat.;

Corodi Integraph—K & E Cat.; Boy's Curve Drawing Integrator—(6:65), Integraph—(46:146); Integraph—(32:66); Ciné Integraph—M.I.T.
Integrometer—Hele-Shaw (31:189); (46:96); Integral Computer—(19:199 or 200)

Part B. Slide Rules

Linear Slide Rule—(52:348); K & E Cat.; Cursors on Rule—(31:170)
Circular Slide Rule—(31:163); K & E Cat.; Pocket Slide Rule—(31:175)
Cylindrical Slide Rule (Thatcher)—(31:174); K & E Cat.
Continuous Slide Rule (Paisley)—(67:4.2—Paisley, 1)

Part C. Solving Equations

With Balances—(32:132)
With Liquids—(32:135)
Isograph—From *Bell Laboratories Record*, Vol. 16, No. 4, December, 1937, pp. 130-140
Differential Analyser—*Encyclopædia Britannica*, 14th Ed., Vol. 4, p. 548
Wilbur Analyser—From M.I.T. News Service
Harmonic Analyser—Michelson and Stratton, (6:74); Michelson and Stratton, (22:149); Henrici, H. A.—(6:71); Henrici—Cuadi—(22:141); H. A. (19:214); H. A. (46:166); H. A. (46:170); Tide Predicting Mach.—(31:251)

Part D. Business Machines

1. Early Methods
    Chinese Bamboo Rods—(54:96) (54:140)
    Korean Bone Computing Rods (56:53)
    Greek Wax Tablet—(56:50)
    Finger Reckoning—(52:77)
    Reglettes de Grenaille—(17:15); (32:8)
    Tally Sticks—(52:26); (55:193)
    Abacus—Salemis (Greek)—(55:162); Roman (Bronze)—(56:51); Use of Roman (67:4.0, Mult.-Div. 2); Chinese (6:10); Modern Chinese—(56:54); Modern Japanese (55:173); Modern Russian (S'Choty)—(55:176); Chinese (57:26); Japanese (57:26); Roman—(57:25)
    Knotted Cords—(11) N. S. American Quipu—(42)
    Napier's Rods—(52:339); (6:11); (67, 4.0 Mult.-Div.—5); Cylindrical Form —(6:12)
    Product Table—Ready Reckoner (67, 4.0, Mult.-Div.—18)
2. Adding Machines
    Pascal (1642)—First Machine with "Carry Principle"—(52:351), (62:11), (2:180), (67:4.0, Mult.-Div.—7)
    Leibnitz (1671)—Stepped Drum Principle—(61:132), (67:4.0, Mult.-Div.—7), (67:4.3, Calc. Mach.—3) (67:4.3—Calc. Mach.—4)

- Thomas de Colner (1820)—First Commercial Mfgr. Arithmometie (2:184); Arithmometer (6:20)
- Felt (1887)—Touch Machines. Principal of Touch Machine (13:11); "Macaroni-Box" Model—(62:52); First Comptometer—(62:56); Comptometer, Early Form (6:27); Comptometer Key-controlled; portion showing mechanism (6:27, 28); Mod. Outside—From Comp.
- Addition-Subtraction Models. Add-Sub (1842) (2:182); Underwood 10-Key Machine—From Comp; +, × Calc. (Burroughs)—From Comp.; +, − Printer (Burroughs)—From Comp.; Bur. Printing Adder (67:3.21, Simp. Reg.—21)
- Special "Plus" Model—Ten register machine (67:3.12; Plus p. 4)

3. Calculating Machines
    - Bollée (1887)—First Direct Multiplier. Machine à Calculer (2:189); Plan (62:186); Principle (67:4.3, Calc. Mach.—16); Principle (17:74), Principle (13:40)
    - Millionaire Machine (1893). Millionaire Calc. Mach. (6:28); Working Diagrams (66:4.3, Calc. Mach.—17); Plates (67:4.3, Calc. Mach.—17)
    - Active and Inactive Pin Principle. Front Elevation (67:4.3, Calc. Mach.—8); Brunsviga Mach. (67:4.3, Calc. Mach.—7); Brunsviga & Interior (6:22); Connection Power Machine (67:4.3, Calc. Mach.—12)
    - Proportional Gear Principle. Diagrams (67:4.3, Calc. Mach.—15); Marchants—(From Comp.)
    - Proportional Rod Principle. Diagram (67:4.3 Calc. Mach.—11); Mercedes-Euklid Calc. Machines (1910) (6:29)
    - Miscellaneous Calculating Machines
        - Moreland (1666)—(6:14, 15)
        - Stanhope (1775-1777). (6:18)
        - Arithmaurel (1854). (2:186)
        - Barbour (1872)—(62:180)
        - Roth (c. 1888)—(2:187)
        - Tchebichef Multiplier (1893). (2:187)
        - Madras (1920). (6:30)
        - Monroe Calc. (1923)—(6:30)
        - Unitas (1925). (6:21)
    - Calculating Machines, *Encyclopædia Britannica*, 14th Ed., Vol. 4, p. 548
4. Advanced Machines
    - Recording and Listing Machines
        - Early Felt (1888)—(62:119)
        - Burroughs Early (1897)—(6:24)
        - Accounting Machines, Underwood—From Comp.
        - Burroughs Bookkeeping—From Comp.
        - Burroughs Typewriter Calculator—From Comp.
        - Burroughs Multiple Form Writer—From Comp.
        - Underwood Double Automatic Feed Billing Mach.—From Comp.

- Statistical Type Machines
  - Remington-Rand Punch—From Comp.
  - Remington-Rand Sorter—From Comp.
  - Remington-Rand Printer—From Comp.
  - Remington-Rand Multiplying Punch—From Comp.
  - Tabulator Card all punched—(20:7)
  - Sorting Brush of Card Tabulator—(20:11)
  - Automatic Plugboard—(20:14)
- Difference Engine
  - Babbages (1812)—(6:32)
  - Scheutz (1858) and Wheelwork—(6:35)
- Miscellaneous Advanced
  - Lightning Cashier—(67:9.11, L. Cal.—1)
  - Burroughs Cash Register (2)—From Comp.
  - Payroll Machine,—International Cat.
  - Pari-Mutuel Betting Machines

GEOMETRICAL TOOLS

Robert C. Yates

THE RULES and tools of the plane geometry of Euclid are too frequently misunderstood. After we grant the postulates and axioms fixed upon by Euclid and Plato, the general question of constructibility becomes important, a question which concerns the nature of the tools and the precise manner in which they are used. Following the dictates of Euclid, we state the rules governing the straightedge and compasses [6, 7]:

(1) The straightedge establishes the straight line of indefinite length through two given distinct points.

(2) The compasses establishes the circle passing through a given point with center at a second given point distinct from the first.

These operations are the only ones permitted. An illustration of their limited nature is afforded by the problem of drawing the bisector of the given angle *AOB*. First select any arbitrary point *P* on *AO* through which is drawn the circle with center at *O*. This meets *OB* in *Q*. Now with center at *Q* draw the circle through *P*, and with center at *P* the circle through *Q*. These meet in *M*, and the line *OM* when drawn is thus the bisector. (See Fig. 1.)

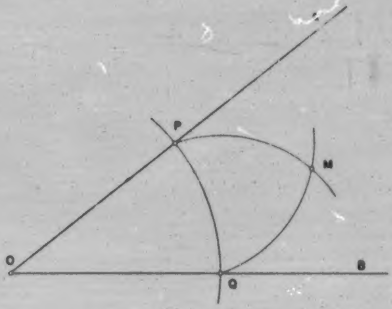

Figure 1

The disagreement here with the usual practice lies in the selection of the circles $P(Q)$ *and* $Q(P)$.[1] The prevalent classroom custom allows arbitrarily drawn circles involving a process of carrying a fixed radius, a practice certainly not granted by the ordinary rules for such work.

The point is again illustrated by the problem of drawing the circle with center at O and given radius AB. Since we are not permitted to carry the radius into position, we must establish a point X such that $OX = AB = r$. This is accomplished as follows. Draw circles $O(A)$ and $A(O)$ meeting in C. Draw the equal line segments OC and AC. Now draw circle $A(B)$ to meet AC in D; then $C(D)$ to meet OC in X, a point on the required circle. This is evident since $AB = AD = r$; $CD = CX = OC - r$. (See Fig. 2.)

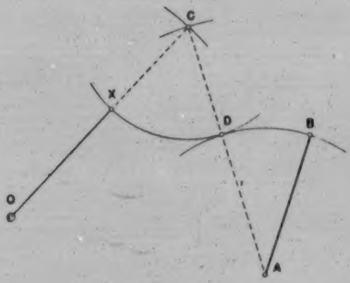

Figure 2

Thus the nature of the Euclidean compasses prescribes *collapsibility*. That is [7], since the compasses does not incorporate the principle of the *dividers,* it apparently folds up automatically when its points are lifted from the plane. Once the demonstration of Figure 2 is made, of course, the privilege of drawing circles whose radii are not given in position is to be assumed. Considerable added interest is attached, however, to problems of construction when processes are firmly restricted to the original rules [11].

Plane geometrical tools other than the straightedge and compasses are generally called non-Euclidean. These include the *parallel* and *angle rulers,* the *carpenter's square,* the *compasses of Hermes,* and various *linkage devices* [11]. We shall discuss a few of the more elementary of these.

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[1] The notation $P(Q)$ signifies the circle through Q with center P.

The *parallel ruler* [2, 4, 11], an instrument of indefinite length, has two parallel straight edges. The width of the ruler may be selected as the unit of length. It is used in a twofold fashion: first, to establish the line through two given points, A and B, and its parallel at a unit distance; second, to determine lines separated by a unit distance passing through each of two given points, A and B, $(AB > 1)$. These operations are illustrated in Figure 3. Note that the positions in the right-hand drawing place in our possession a rhombus and thus two perpendicular lines, an item of considerable importance.

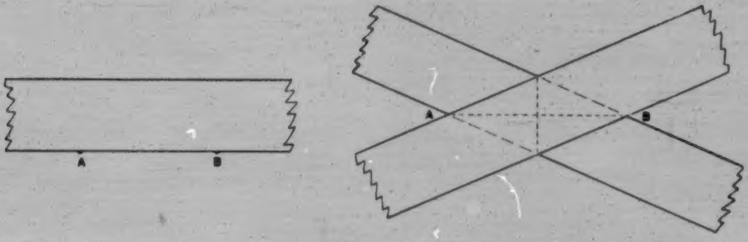

Figure 3

It is not difficult to prove that the parallel ruler is equivalent to the compasses and thus is sufficient equipment to make all plane constructions of a Euclidean nature [11]. As an illustration, consider the bisection of an angle AOB. Place the ruler first with one edge along OA, then along OB. Lines internal to the angle determined by the other sides of the ruler locate a point P on the bisector, as in the drawing at the left in Figure 4.

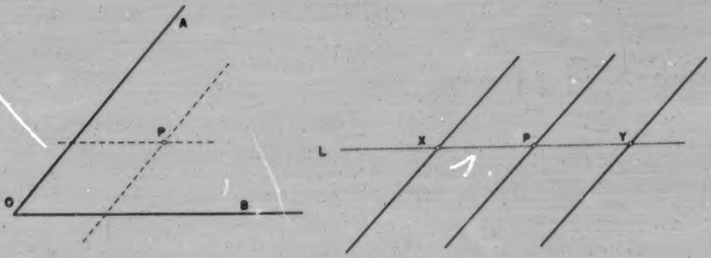

Figure 4

The perpendicular to a line *L* at a point *P* on *L* is erected as follows. Place the ruler in an arbitrary position with one edge through *P*. The other edge cuts *L* in *X*. Now turn the ruler over to establish a third parallel line and another point *Y* on *L*. If the ruler be placed with its edges through *X* and *Y* in its two positions, a rhombus is formed with one diagonal as *XY* and the other as the required perpendicular.

The general-*angle ruler* [2, 4, 11, 12], of which the parallel ruler is a special case, is also capable of effecting all constructions of classical plane geometry. To illustrate its use, construct the parallel to a line *L* through a point *P*. Place the ruler with one edge along *L* and the other through *P*. Then replace the ruler parallel to its original position so that the second edge passes through *P*.

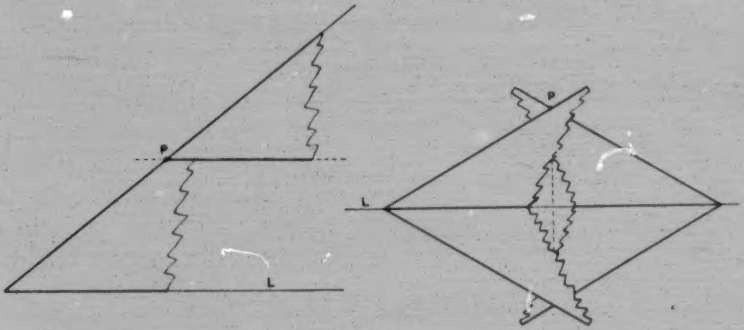

Figure 5

The perpendicular from *P* to *L* is obtained by placing the ruler in two opposite positions on one side of *L* and then reflecting in *L*, as in Figure 5. These four positions determine a rhombus, one of whose diagonals is perpendicular to *L*.

An interesting construction, which we leave here as entertainment, is the location of arbitrary points on a circle with center *O* and radius *OA* [11].

We have alluded several times to the fact that all Euclidean plane constructions[2] may be accomplished by the compasses alone [1, 3, 4, 5, 7, 8, 9, 11]. This fact was established by Georg Mohr and Mascheroni in the late seventeenth and eighteenth centuries

[2] With the exception of the formation of a *continuous* straight line.

with a theorem that created considerable interest among geometers. The proof of its equivalence to the straightedge-compasses combination reaches its zenith in showing that the point of intersection of two lines, given only by two pairs of points, can be located by the compasses without recourse to the straightedge. Unfortunately, space does not permit the demonstration here.

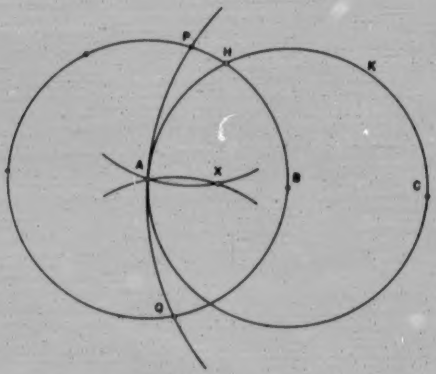

Figure 6

A beautiful construction illustrating this so-called geometry of Mascheroni is the location of the mid-point of the segment AB (Fig. 6). Construct the circle $B(A)$. Now with the same radius, locate the hexagonal points $AHKC$, where C is collinear with A and B and diagonally opposite A. Let $C(A)$ cut $A(B)$ in P and Q. Finally, $P(A)$ and $Q(A)$ intersect in X, the mid-point of AB. For, if $AB = PA = QA = r$, then $AC = CP = CQ = 2r$; and, since PAX and CAP are similar isosceles triangles,

$$AP/AX = PC/AP = 2.$$

Accordingly,

$$AX = (AP)/2 = (AB)/2.$$

Almost all standard constructions offer special interest when performed by the compasses alone. This tool, incidentally, is the natural medium through which the geometry of *inversion* is executed.

Although quite useful as an auxiliary instrument in general plane constructions, the ***straightedge*** certainly does not appear very powerful. Surprising, however, is the fact that it is capable of solving elaborate and complicated problems of construction. An example of this is the remarkable construction of the tangents to any given conic from an external point. As an illustration, the tangents to a circle from an external point *P* are located by first drawing three arbitrary secants from *P*. By cross-joining the points of intersection of these secants with the circle as shown, the line *XY* which cuts the circle in the points of tangency is determined, as in Figure 7.

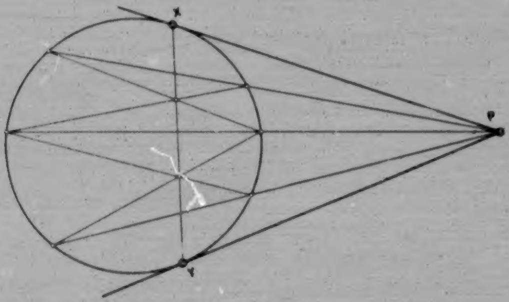

Figure 7

A field in which the straightedge plays a natural role is that of projective geometry. (The foregoing construction may be recognized as a special application of Pascal's theorem on poles and polars.) It is here that lines and their intersections are dominant features, while such notions as distance, angle, area, parallelism, and the like have no interpretation whatever.

The straightedge becomes considerably more effective if somewhere in the working plane there is given either a circle with center, a square, a parallelogram, a conic, or some other identified configuration. Such arrangements produce systems known as the geometry of Poncelet-Steiner.

Fundamental in these systems is the construction through a given point *P* of the line parallel to a bisected segment *AOB*. Draw *PB* and *AP*, and upon the latter select an arbitrary point *E*. Draw *EO* intersecting *PB* at *I*. Then *AI* meets *BE* in *Q* such that PQ is parallel to AB. This may be recognized as the quadrilateral construction of harmonic points or, in more elementary fashion, as the construction of a diameter PQ of a circle homothetic to another with diameter AOB having centers of similitude at I and E.

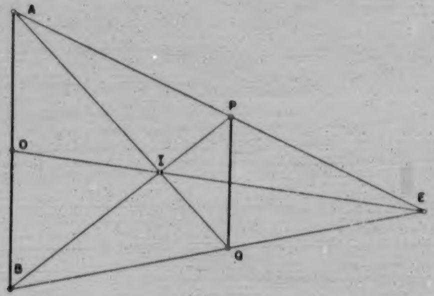

Figure 8

A few exercises may serve to awaken an interest in this type of construction:

(1) Upon a fixed circle with center O, draw the diameter that is parallel to a given line L. (Establish a bisected segment upon L by drawing three parallel lines of arbitrary direction, one of which passes through O.)

(2) Given a circle with center O. From a point P construct the perpendicular to a given line L.

(3) Given a square. Draw the line through a corner parallel to a diagonal. (The diagonals yield the center of the square and Figure 8 applies.)

(4) Bisect the sides of a given square.

The preceding discussion forms but the briefest introduction to the general subject of mathematical tools. But it is through the study of the nature of these instruments that a working knowledge of plane geometry and a sympathy with its structures are acquired.

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MATHEMATICAL APPARATUS

Phillip S. Jones

IN THE CLASSROOM, mathematics club, assembly program, or exhibit, mathematical demonstrations which use moving apparatus as well as visual diagrams and verbal explanations attract and retain more attention and show relationships more clearly than do static displays. This is particularly true if the apparatus calls attention to the appearance or use of mathematics in daily life.

In Figure 1, for instance, the pin and string construction for an ellipse, the setup for which is seen in the upper left-hand corner, gives a mechanical construction for a curve which is continually seen about us. This construction serves not only to visualize a locus definition, namely, that the ellipse is the locus of a point moving so that the sum of its distances from two fixed points is constant, but it also explains why the elliptic "gears"[*] at the bottom of the board will mesh. For these congruent ellipses are pivoted at their foci with the distance between these foci equal to the constant sum of the focal radii; hence the radii to the point of contact may vary, but their sum remains constant. These gears convert uniform rotary motion into variable rotary motion and are useful, for example, in such machines as slotters and shapers where a slow working stroke and a quick return stroke are desired.

In constructing these "gears" from plywood, it was found necessary to glue felt to their edges to provide traction and to join the two foci not used as pivots with a link equal in length to the major axis in order to prevent the gears from falling apart at some points of their cycle. This same motion can be obtained with a simple four-bar linkage, as shown in the upper right-hand corner of Figure 1. This linkage offers many opportunities to show interesting geometric relationships and to give practice in functional

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[*] The selected references at the end of this section are classified to follow the order of development in the body of the paper and therefore will not be referred to for each individual topic.

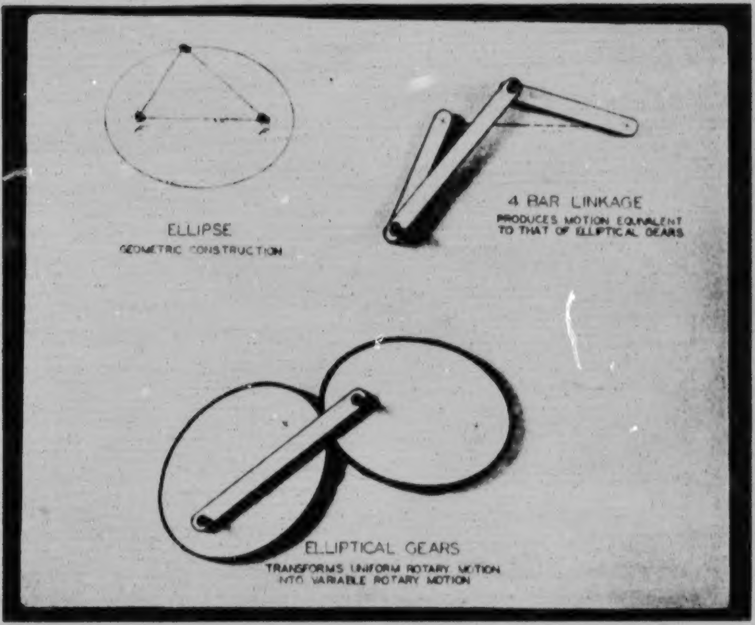

ELLIPSE
GEOMETRIC CONSTRUCTION
4 BAR LINKAGE
PRODUCES MOTION EQUIVALENT TO THAT OF ELLIPTICAL GEARS
ELLIPTICAL GEARS
TRANSFORMS UNIFORM ROTARY MOTION INTO VARIABLE ROTARY MOTION

Figure 1

thinking. If you let the long bars (one "bar" of the linkage is supplied by the board on which it is mounted) become uncrossed, you have a parallelogram which can be used for transmitting uniform rotary motion; if you distort the parallelogram again, you have an apparatus for changing rotary motion into reciprocal motion. If you generalize this four-bar linkage a little more by not requiring the opposite sides to be equal, the immediate problem (Given the relative lengths of the links, what kinds of motion are possible?) leads to much real functional geometric thinking in a situation that has many practical applications.

The property of the ellipse demonstrated by this pin and string construction readily explains why naval pilots, flying out from an aircraft carrier and planning to return to another base or to the same carrier in another position, find, if there is no wind, the territory over which they may fly bounded by an ellipse. This is one of the so-called "radius of action" problems which a former student characterized in a letter as being most interesting because the problem is to "find the carrier or you're a dead pigeon."

Of course, ellipses are most commonly seen as the apparent shape of circles viewed at an angle. The draftsmen meet them so often that many drafting rooms have elliptic trammels or ellipsographs more precise than that shown in Figure 2, but operating on the same principle. This ellipsograph may be fastened to a blackboard with suction cups; the knobs make it possible to draw ellipses of various shapes by setting any desired semi-major and semi-minor axes.

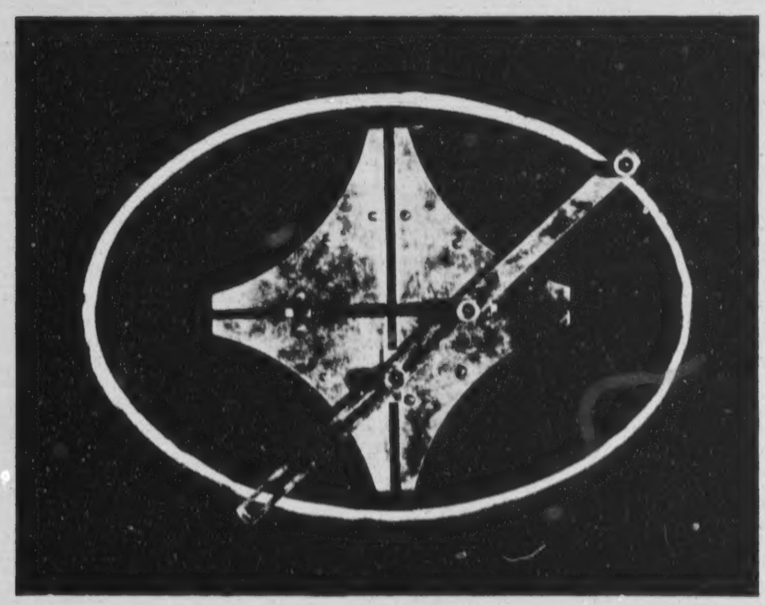


Figure 2

The student who made and demonstrated the elliptic gears found a piece of old brass pipe, cut it at an angle, had it plated, and then had a grand time at a public exhibit explaining to passers-by that everyone knew that salami was always cut at an angle, thus giving elliptic slices, because such slices were the shape that best fitted into sandwiches—another application of the ellipse.

An elliptic "billiard board" is pictured and described in another article on "Mathematical Demonstrations and Exhibits" (p. 90), as are also the paraboloid formed by the surface of a rotated fluid, models of parabolic and catenoidal cables, and the demonstration of the catenary as the sail curve. The parabola and the catenary look so much alike that persons first viewing them will need convincing that they are not the same; but a parabolic template (Fig. 3) with a hole drilled at its focus for a piece of chalk will enable one to draw a catenary, the center curve, by rolling the parabola along the meter stick which is held by two suction cups. A piece of string can be held up and shown to fit the curve of this catenary but not that drawn around the parabolic template. The triangle, string, and suction cup shown at the extreme right in Figure 3 provide a method of introducing and drawing the parabola as the locus of points equidistant from a fixed point (the focus, represented by the suction cup) and a fixed line (the directrix, represented by the meter stick). A piece

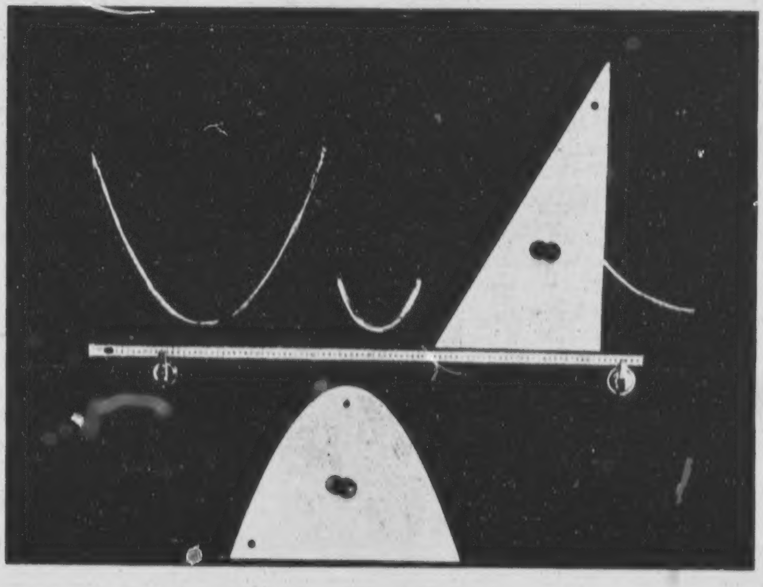


Figure 3

of chalk held against the triangle is forced by the string, which is fastened to the upper vertex of the triangle and to the suction cup, to slide upward along the triangle as the triangle is slid along the straightedge. The chalk traces the parabola.

Possibly the most common locus seen today is that of a piece of mud on the tire of an automobile. If the mud is on the extreme outside edge of the tire, it traces the cycloid shown in Figure 4. This curve has so many interesting properties, so many famous names (Galileo, Newton, Leibnitz, Bernoulli) associated with it, and has featured in so many famous quarrels that it has been called the Helen of Geometers, for Helen of Troy. The tracks at the right in the picture demonstrate two of its most interesting properties: namely, that it is, under the action of gravity, the path of quickest descent, the brachistochrone, from one given point to another in a vertical plane, as shown by the fact that, when a ball on the cycloidal and one on the linear path are released at the same time, the former arrives at the bottom first. It is also the tautochrone or curve of equal time, as shown by the fact that, if

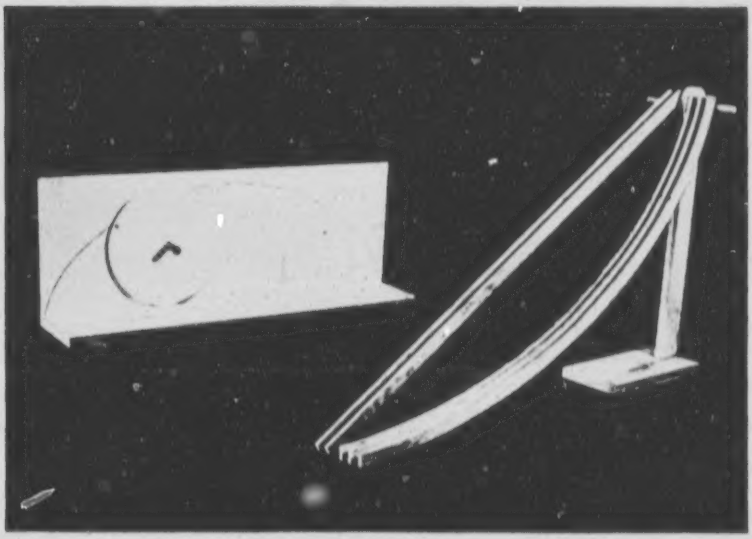


Figure 4

one ball be released from the top of one cycloidal track and one from the middle of the other, they both arrive at the bottom at the same time. Scientists have tried to use this property in designing clocks.

Plywood forms, such as those shown in Figure 5, stuck to a blackboard with suction cups, and a piece of chalk fastened to a cereal box which will roll on the plywood forms facilitate the demonstration of the locus of a circle rolling on another circle; namely, the epicycloid, which played its part in the Ptolemaic theory of the universe and which, with the cycloid and hypocycloid, forms the curves used in the design of cycloidal gear teeth and gear trains. The curve at the left is the two-cusped epicycloid or nephroid and that in the center is the cardioid, named for its heart-like shape. These two curves can also be demonstrated as caustics. The nephroid is the envelope of light rays reflected from a circular water glass or napkin ring for a light source emitting parallel rays, while the cardioid is the reflection pattern for a point source of light on the circle.

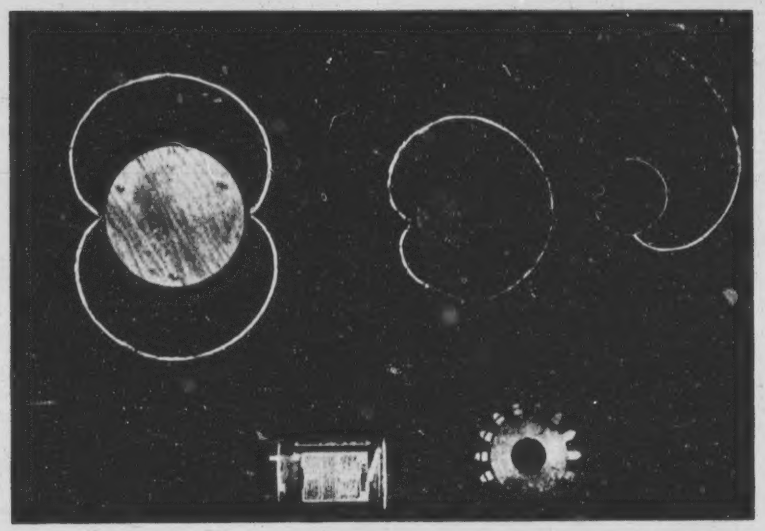


Figure 5

The cycloid is the locus of a point on a circle which rolls with-

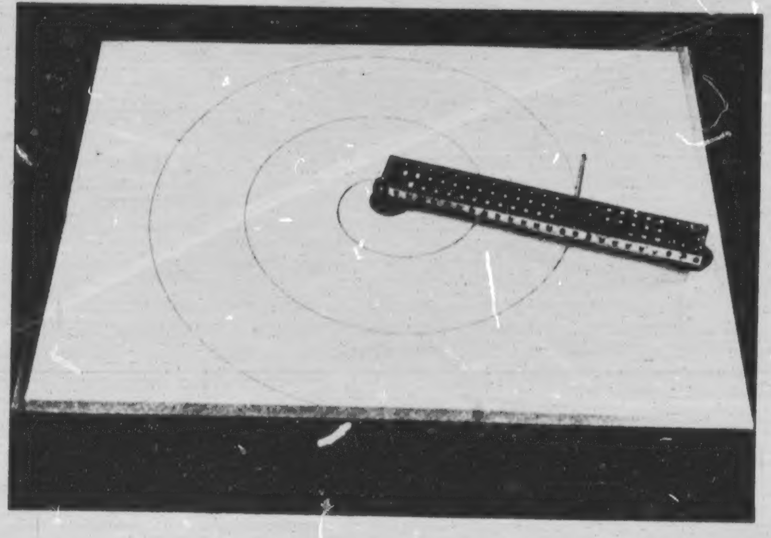
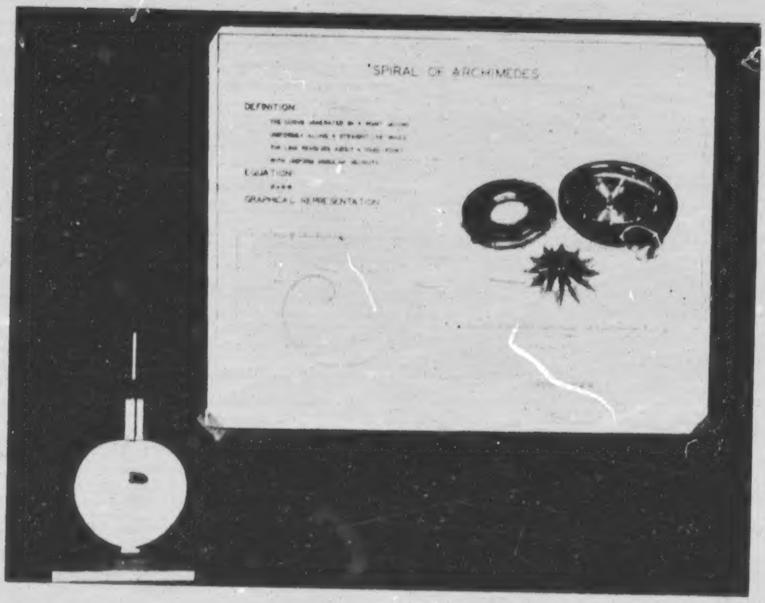


Figure 6



SPIRAL OF ARCHIMEDES
DEFINITION
EQUATION
GRAPHICAL REPRESENTATION

Figure 7

out slipping on a straight line. If a straight line rolls on a circle, the curve is that seen at the extreme right in Figure 5, which was traced by a piece of chalk on the end of a string as the string was unwound from a cereal box held with its base against the blackboard. This curve is the involute of a circle, and is used commonly today in the design of gear teeth, for which purpose it is superseding the cycloids. It is also used in designing some cams.

A fly, represented by the upright peg or pencil in Figure 6, walking out on the spoke of a uniformly rotating wheel, represented by the slotted arm mounted on a fixed vertical axis, would, as seen in the figure, trace a spiral of Archimedes. The movement of the peg in the apparatus in Figure 6 is accomplished by fixing it to a slider moving in the slotted arm. This slider is in turn fastened to one side of a string belt running around a fixed pulley mounted on the vertical axis of rotation and a second idler pulley at the end of the rotating arm. This spiral has not only an interesting history, as its name implies, but several modern uses. The cam at the lower left in Figure 7, each side of which is shaped as an Archimedean spiral, converts uniform rotary motion into uniform reciprocating motion. The casings of centrifugal pumps, such as the German supercharger shown in the photograph, follow this spiral to allow the air which increases uniformly in volume with each degree of rotation of the fan blades to be conducted to the outlet without creating back-pressure. H. T. Brown in his "507 Mechanical Movements" suggests the use of this curve as a guide for the feed motion in a drilling machine.

The apparatus in Figure 8 consists of a base circle, a point on the circle represented by a suction cup, a straight line represented by a stick which is constrained by a bracket on the suction cup to pass always through the fixed point, and two nails representing two points on the line a fixed distance apart. If the first nail follows the fixed circle, the second nail, or a piece of chalk substituted for it, traces out the limacon or snail curve of Pascal. This curve, when used as the shape of a cam as shown at the left in Figure 9, converts uniform rotary motion into simple harmonic motion. The device at the right in Figure 9 can be thought of as proof of this fact because the two followers can be seen to move together, and the second device, which is a Scotch crosshead, is

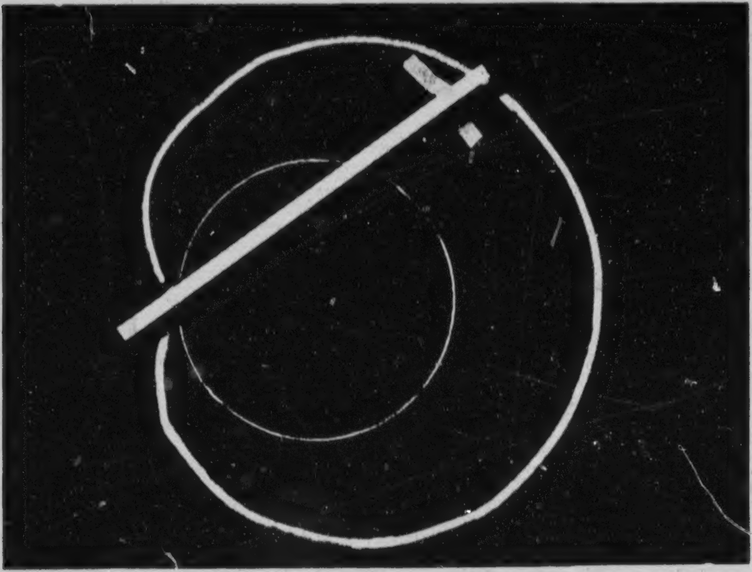


Figure 8

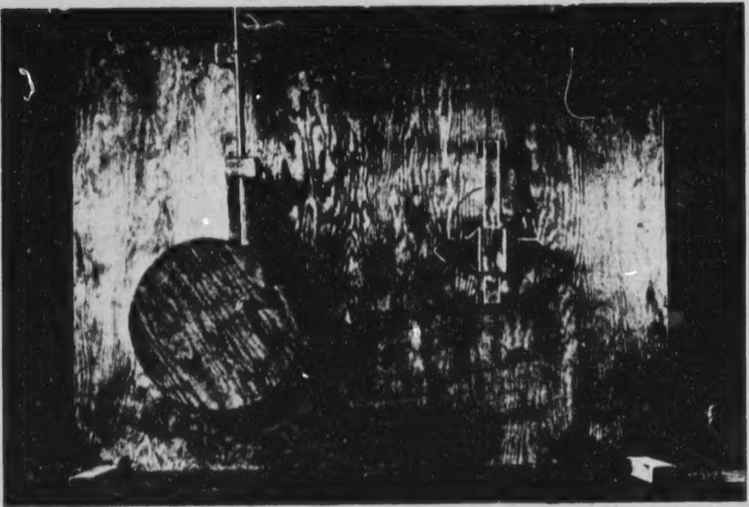


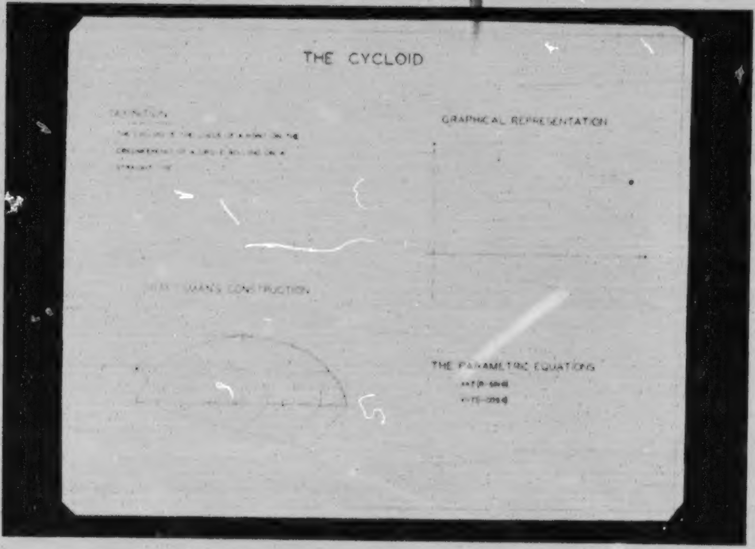
Figure 9

constructed according to the definition of simple harmonic motion as the motion of the projection, represented by the crosshead, of a point traveling at a uniform rate on a circle, represented by a rotating arm which has a peg sliding in a slot on the back of the crosshead. A pencil fastened to this follower would draw a sine wave on a roll of paper drawn past it.

Incidentally, the cardioid, or heart-shaped curve, is also a special case of the limacon and may then be demonstrated by the method suggested.

Much interesting and simple trigonometry may be shown to apply to models of screw threads and gears. The history and logic of logarithms and the slide rule can be made more real by demonstrations of Napier's bones and Gunter's scale, and trisection devices always involve good geometry and have a never-ending appeal. These items and calculating machines and planimeters have been mentioned in a previous article (see pp. 94 to 96). Many other interesting devices will be found suggested and described in the general sources listed on page 223.

The equipment which is pictured here was made and demonstrated at the Edison Institute of Technology, Dearborn, Michigan, in connection with classwork and exhibits by freshman and sophomore college students. However, the conic sections fit in well with the work done in functions and graphical representation in many second year high school algebra courses. Constructions, such as those shown on the charts in Figures 10 and 11, exist for all the curves discussed and are so varied and simple, as well as useful in the drafting room, that high school geometry students could easily do them and could see real use made of constructions for tangents, formulas for lengths of circular arcs, and so on. Even junior high school students could draw many of these curves in connection with work done in learning the use of instruments. Certainly it seems that quadratic equations should never be left without a look at their graphical solution and some information about the properties of the related parabola. The other simple loci which were discussed earlier can add much that is practical and attention-getting, as well as thought-provoking and functional to the study of loci. These loci are more real in their use than and as simple in their construction as those problems about buried



THE CYCLOID

DEFINITION

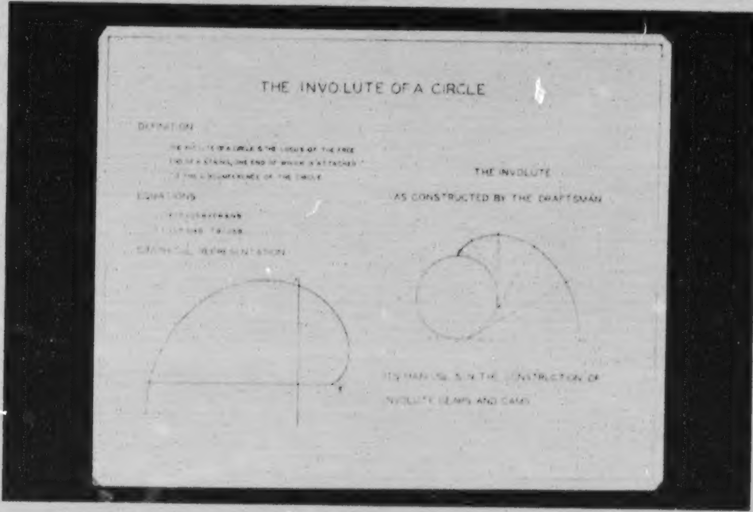
THE CYCLOID IS THE LOCUS OF A POINT ON THE CIRCUMFERENCE OF A CIRCLE ROLLING ON A STRAIGHT LINE

GRAPHICAL REPRESENTATION

DRAFTSMAN'S CONSTRUCTION

THE PARAMETRIC EQUATIONS

Figure 10



THE INVOLUTE OF A CIRCLE

DEFINITION

THE INVOLUTE OF A CIRCLE IS THE LOCUS OF THE FREE END OF A STRING, ONE END OF WHICH IS ATTACHED TO THE CIRCUMFERENCE OF THE CIRCLE

EQUATIONS

GRAPHICAL REPRESENTATION

THE INVOLUTE AS CONSTRUCTED BY THE DRAFTSMAN

ITS MAIN USE IS IN THE CONSTRUCTION OF INVOLUTE GEARS AND CAMS

Figure 11

treasure which are found in the few paragraphs on loci included in many geometry textbooks.

Apparatus of all sorts is useful in the classroom in giving the student added insight into and practice in seeing and thinking about mathematical relationships, as well as in getting his interest and attention. The students who design and make these "gadgets" after studying their history and use are, of course, the ones who get the most real fun and profit from them. However, these devices accompanied by charts and explanations will attract attention and bring questions and interested, understanding comment not only from students in the classroom and mathematics club, but from passers-by at school open-house exhibits or assemblies.

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MATERIALS FOR MATHEMATICAL MODELS: THEIR SELECTION AND USE

Joseph Hilsenrath

An experienced modelmaker, whether he be a builder of airplane models, ship models, model railways, model racing yachts, or mathematical models, as long as he is an enthusiast, has many of the following characteristics: he never throws anything away and so has an accumulation of junk around his house; he stops to look at any and every window he passes and hence seldom arrives home on time; he is a fairly good conversationalist and thus gains a lot of free advice and information from others; he is a lover of machinery and is forever buying some new gadgets and then doing without lunch once or twice a week; and he is constantly on the lookout for some ready-made article which he can with little effort appropriate to his needs and which will result in a decided economy in both time and money in the making of a future model.

A lesson which every teacher can learn from this enthusiast is to be on the alert for materials which can be employed in or adapted to the production of models and equipment for the classroom. Some of the raw materials which can be used for this purpose may be waste material in some industry. Ingenuity is the teacher's best assistant. Articles manufactured for other purposes or for general consumption may often be adapted to instructional purposes. The most expensive material is that which has only a special function and hence is in demand by only a limited number of users.

An examination of commercial catalogues and publications will give many clues to available materials. These may include catalogues from modelmakers, supply houses, toy manufacturers, commercial steel dealers, hardware dealers, stationery houses, mail-order lumber yards, scientific instrument companies, photographers' suppliers, automobile supply stores, wholesale radio suppliers, window trimmers' supply houses, as well as trade magazines and hobby publications.

A leisurely trip through a five-and-ten-cent store or a large hardware house will give an idea of articles which can be used or easily adapted to model needs. Spheres may be found as croquet balls, billiard balls, marbles, rubber balls, tennis balls, spherical balloons, beads, and various other ornaments. Cylinders, cubes, and prisms appear as candy boxes, tin cans, powder boxes, food cartons, mailing tubes, and toys. Cardboard or metal discs may appear as mats or may have been discarded as waste by stamping industries. Wire circles are used by manufacturers of lamp shades. Some of the small solids are made by toy manufacturers, and some of the larger ones are often used by window display firms. Gears, pulleys, circles, and links from such sets as Meccano or Erector can also be used to advantage.

Models can be made and used by student and teacher alike. Some models should be constructed by every student in the class. They need not be elaborate and may be restricted to those which can be made from cardboard or string. They may be slipped into an envelope-pocket in the student's notebook and can be used effectively in developing intuitive and informal work. The teacher may find it advantageous to produce some models in front of the class with the aid of a few pieces of cardboard, some strips of wood of various lengths, a razor blade mounted in a handle, a paper punch, and some brass paper fasteners. Other models can be carefully made and kept available, ready for class demonstrations at a moment's notice.

As an example of a model which can be made by students or teachers under various conditions we may consider the cube. A cube may be made in all of the following ways:

(1) Twelve toothpicks glued together at the ends to form the edges.

(2) Twelve wooden applicators glued as noted above, but painted.

(3) Twelve pieces of stiff wire soldered at the vertices and painted.

(4) A paper or cardboard template folded and edges pasted together.

(5) Six squares cut from stiff celluloid or transparent plastic material and glued together at the edges.
(6) Solid cube cut from soap.
(7) Solid cube cut from modeling clay.
(8) Solid cube cut from a block of wood.
(9) Solid cube cut from aluminum, brass, or plastic (opaque or transparent).
(10) Sheet metal folded and soldered.

Numbers 1, 2, 4, 6, and 7 are the simplest to make, require few tools, and can be constructed by everyone in the class. The cost of 1 and 4 is nil. Numbers 6 and 7 may be transformed into a number of semi-regular solids by cutting off the proper corners. Solids constructed of brass or other metals require little machining and are excellent for display purposes.

The nature and number of materials and tools required will grow along with the need and use made of models. The following list may prove suggestive.

Materials	*Tools Required*	*Tools Recommended*
1. Toothpicks	None	
2. Wooden applicators	A*	
3. Venetian blind slats	A, B	Small drill
4. Steel strapping	G, H, I, J, L	
5. Cardboard	A, B, C	D, E
6. Bus-bar wire	G, K	
7. Thin florist's wire	G	
8. Steel wire circles	G, K	
9. Plywood	M	N
10. Sheet metal	H, K, L	
11. Brass strips and bars	H, I, O	Small drills
12. Modelmaker's lumber	A, F, M, N	
13. Balsa wood	A	
14. Glass	P	
15. Soap	A	
16. Clay	A	
17. Colored string	None	
18. Plaster of paris	None	
19. Plastic sheets	A	
20. Plastic tube, bar, and rod	A, M, N	

* See the following page for key to letters.

Tools

A. Razor blade type knife[1]
B. Paper punch
C. Straightedge
D. Pair of scissors
E. Small hammer
F. Stapler
G. Pair of side-cutting pliers
H. Scriber
I. Metal punch[2]
J. Eyelet punch
K. Soldering iron
L. Metal shears
M. Coping saw
N. Crosscut saw
O. Hack saw
P. Glass cutter

Accessories

a. Paper fasteners
b. Eyelets
c. Staples
d. Paste, glue, airplane cement
e. Rubber bands
f. Elastic, round and flat
g. Wire brads
h. Suction cups
i. Solder and soldering paste
j. Tape, masking, drafting, binding, etc.
k. Violin D-string
l. Thumb tacks
m. Gummed letters and figures
n. Paints, assorted colors
o. Leather strips
p. Golf tees

Since some of the items listed above may appear to be out of place, a few of their advantages will be illustrated. Paper fasteners are convenient pivots for hastily constructed models made of cardboard or wood strips. If more durable and accurate pivots are required for cardboard or steel strips, eyelets should be used. An accurate, inexpensive pivot may be made from a piece of gut or D-string, as follows: (a) pierce the two pieces of cardboard with a needle of suitable diameter and with a knife cut a piece of gut about an eighth of an inch longer than the thickness of the two pieces to be pivoted; (b) insert the piece of gut into the holes and touch a heated object to each end in succession in order to spread the ends and form a perfect rivet which will resist the wear of cardboard, fiber, or plastics.

Rubber bands and round elastics are ideal for setting up geometric figures on theorem boards and may be used in making a variable model of a ruled surface where it is necessary that the reguli are always straight lines. Suction cups may be used to fasten models to the blackboard; some cups are available with a bolt which permits pivoting. Tape is valuable in holding materials in place during gluing, as a masking guide for painting, and as a binder for models made of glass. (The writer has found Kodak Binding Tape to be most satisfactory; 10 yards of ⅜″ width cost 20¢.) Leather strips may serve as hinges for plywood models illustrating areas. Golf tees are inserted in the holes of theorem boards and the elastic is attached to them. On the broad surface or head, small letters may be pasted to identify various points. Gummed letters and figures will add to the appearance of a model and save time in its construction.[3]

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[1] There are many such knives on the market priced from 10¢ to $1.00. The writer recommends the "Skyway Utility Knife," Skyway Model Aircraft Supply Co., 426 Sixth Ave., Brooklyn, N. Y.; 10¢, with a special double-edged blade. Spare blades are two for 10¢.

[2] A satisfactory punch costs about 75¢. It may be obtained in a number of sizes from one-sixteenth inch to one-quarter inch. If a number of sizes of holes is desired, a larger punch with replaceable dies sells for about $1.00 or $5.00. Either of these will cut a hole in a piece of metal one-eighth inch thick.

A brightly painted model has distinct pedagogical value, but it is important that the painting be carefully done because a model is often made or ruined by the way the paint is applied. Bright colors should be selected. A fast drying lacquer should be used; it is most economical to purchase lacquer in tiny bottles often sold in hobby stores for five or ten cents. (Evaporation makes the larger bottles wasteful.)

Balsa wood is the material preferred by some modelmakers. For some models it is quite satisfactory; its disadvantage is its lack of strength. A hobby or neighborhood hardware store may carry various sizes priced from 1¢ to 9¢ each. The following sized strips in 36″ lengths were found in a typical store: 1/32″ x 2″, 1/32″ x 3″, 1/16″ x 1/16″, 1/16″ x ⅛″, 1/16″ x ¼″, 1/16″ x 2″, 1/16″ x 3″, ⅛″ x ⅛″, ⅛″ x ¼″, ⅛″ x 2″, ⅛″ x 3″, ¼″ x ¼″.

Cardboard may be obtained from stationers, printers, bookbinders, and paper supply houses in assorted colors, various thicknesses, and many degrees of rigidity. Some teachers have found the cardboard used by laundries adaptable for models. The only tools necessary to work this material are a straightedge, a sharp razor-blade knife, and paste. To fold the cardboard, score (cut halfway through) it along a straight line with a sharp razor-blade knife and fold away from the side scored. If the blade is quite sharp, it may be drawn lightly along the line, for the weight of the knife alone may cut to the proper depth. A light touch will score even the thinnest cardboard. Pliable cardboard thus cut gives the template an excellent hinge. The writer has a set of collapsible cardboard models with such cuts for hinges; these have been subjected to a year's service without marked signs of wear. Hinges may be reinforced by applying binding tape along the folding edges. If the size of the model necessitates the use of more than one sheet or if one wishes to economize by using small pieces, these parts should be cut to the proper shape, a hinged flap included on every edge, and these flaps either pasted or stapled to their adjoining pieces. Cutting a piece of cardboard requires marking a pencil line on it and cutting along this freehand rather than by means of a guide. Models made from cardboard or paper have included plane figures, planes in space, solids bounded by planes, developable surfaces, some ruled surfaces, solids built from a section of plane curves, and surface linkages.[4]

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[3] The Tablet and Ticket Company, 115 East 23rd St., New York City, makes Willson's gummed paper letters and figures in assorted colors and in 40 sizes from ¼ inch to 4 inches.

Where it is necessary to emphasize lines rather than surfaces, sticks are used to great advantage. Wood applicators, which may be purchased in any drug store, are the best source. They consist of round sticks about 1/16″ in diameter and six and one-half inches in length. They may be purchased in small quantities or by the box at approximately 75 cents for 72 dozen. They can be cut to size with a razor blade or knife, are uniform in thickness, can be joined with airplane cement or collodion, and when painted cannot be distinguished from wire. A model made from applicators is light, transparent, neat, and easily constructed. To provide a more rigid structure diagonals may be placed in faces which are polygons.

Applicators should be painted before the model is started. A single coat of lacquer applied with a small camel's hair brush will be quite satisfactory. No special skill is required and contrasting colors will provide pleasing results. The stick should be held between the fingertips and revolved until it is painted to within one-half inch of each end. One end may then be placed in a piece of clay to a depth of ¼ inch and the upper part painted, leaving a small unpainted portion which is seldom used. Applicators have been used to great advantage in making regular or Platonic solids, semi-regular or Archimedean solids, star polyhedra, some developable surfaces, and some ruled surfaces.

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[4] **Goldberg, Michael, "Polyhedral Linkages." *National Mathematics Magazine*, Vol. 16, pp. 323-332, April, 1942.**

Wire may be used in place of applicators, particularly for those cases requiring circular or irregular shapes, such as helices, surfaces of revolution, and quadric surfaces. The best wire for models is a tin-coated copper wire used in radio work and called bus-bar. It is either round or square, and may be cut readily with pliers, bent into convenient shapes, and easily soldered. Where wire circles are needed, as in the construction of an earth model for map projection, steel circles used by manufacturers of lamp shades are a good source and may be purchased in various sizes in department stores. These steel circles are not readily soldered, but a little experience makes one adept at this work.

Wood strips are important in the construction of classroom models. With the aid of a set of strips of varied length, with holes punched at the end, and a few paper fasteners as pivots, it is possible to make up triangles or quadrilaterals to illustrate the definitions and propositions of Book I and Book III of Euclidean geometry. Strips should be about one-half inch wide and painted. Venetian blind slats, an excellent source of supply, are one-eighth inch thick and either two or two and three-eighths inches wide, and come in lengths exceeding the usual need. Any manufacturer or dealer will give away or sell cheaply discarded slats if he knows their intended purpose. A small manufacturer will probably rip them into one-half inch strips, or the school woodwork shop can perform the same service. Old slats are also satisfactory, a slat two or three feet long costing about a nickel. Ten slats should be more than ample for ordinary requirements. A paper punch may be used to make holes in the ends of the slat, and if the slats are painted in white or ivory, they show up well when held against the blackboard. The slats may also be used in constructing blackboard linkages[5] which may be attached to the board by means of suction cups.

Steel strapping is readily bent into circular shapes and can be used in place of wood strips. It is used by manufacturers' shipping rooms to bind boxes and bales and is valuable in making models of spheres for solid geometry, astronomy, and navigation. It may be secured in either a galvanized or a black japanned finish and in a number of thicknesses and widths. For small spheres and linkages, 3/8″ x .015″ and for larger spheres either 1/2″ or 5/8″ widths are recommended.[6] The strapping should be cut with small metal shears, allowing enough room for the punching of holes with the metal punch described in a previous footnote. Pieces of strapping are joined together with eyelets fastened by means of an eyelet punch. Before being punched, holes should be marked with a scriber. A little practice will make one adept at setting the punch over exactly the required place.

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[5] Hilsenrath, J., "Linkages." *The Mathematics Teacher*, Vol. 30, pp. 277-284, October, 1937.

Transparent models are often desired for display purposes and can be made of glass. It is best to determine the shapes and sizes needed and to have these cut by a glazier. One can do his own cutting, but it is advisable not to try before observing a glazier at his work. Some of the prices quoted for glass are: single thickness, 15¢ per square foot; double thickness, 20¢ per square foot; opaque (opal), 35¢ per square foot; colored (cathedral), 50¢ to 60¢ per square foot; 1/4-inch plate, 55¢ per square foot; grinding, 1¢ per inch; polishing, 2¢ per inch.

Plastics are also useful for transparent models because they can be readily handled, are unbreakable, and present a good appearance. Their disadvantages are that they scratch easily and are expensive. Trade names for transparent plastics are Pyralin, Lumerith, and Plexiglas. The latter is a thermo-plastic and, after being immersed in boiling water, may be shaped into irregular contours. Sheets of plastics come in thicknesses varying from .060″ to .500″ and in sizes ranging from 24″ x 36″ to 36″ x 48″. Round bars come in diameters increasing by 1/8″ from 1/4″ to 2 1/2″. Thin sheets may cost $1.00 per square foot and sheets 1/2″ thick $6.00 per square foot. Less expensive pieces are the thinner celluloid or acetate sheets. Heavy sheets or bars must be cut with a saw and can be machined just as well as any metal. Thin sheets can be cut with a knife. When making a straight cut, guide the knife with a steel or metal ruler and make a cut in the material. This first cut will seldom ever penetrate to the other side; after a slight cut has been made, the piece can be bent back and will break evenly along the cut.[7] The best way to cement plastic material is to dissolve some of it in a solvent, such as acetone, and use the mixture as a cement. This cement is also supplied ready-made by the manufacturer of plastics.

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[6] Stanley Eversafe Japanned Steel Strapping may be obtained locally or from the Stanley Works, New Britain, Conn.

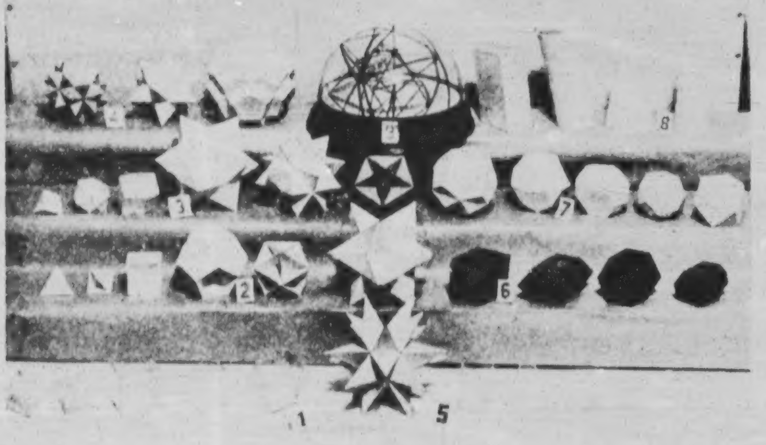

Figure 1

1. Regular Polyhedrons (transparent plastic)
2. Intraverted Polyhedrons (paper)
3. Extraverted Polyhedrons (paper)
4. Stellated Polyhedrons (paper)
5. Intersecting Solids (transparent plastic)
6. Crystal Models (paper)
7. Archimedean Solids (paper)
8. Generated Solids (wood)
9. Model for Euler's Theorem ($f + v = e + 2$) (glass and wire)

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[7] On some devices a hairline is desired. It may be obtained by guiding a knife as before, cutting along the surface, and rubbing India ink into this cut. The result is a perfect hairline. The cut portion should be adjacent to the scale on which it will be used in order to avoid error caused by parallax.

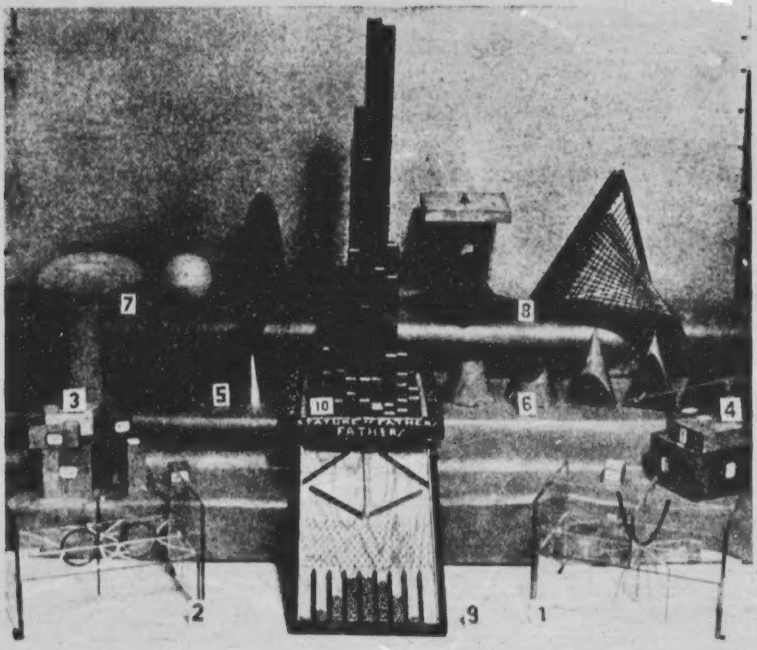


Figure 2

1. & 2. Graphic Representation of Complex Roots (glass)
3. & 4. Models for $(a \pm b)^3$ (wood)
5. Hyperboloid of Revolution (wire) and Cone (brass)
6. Conic Sections (plaster of Paris)
7. Solids of Revolution (wood)
8. Ruled Surfaces (round elastics)
9. Galton Probability Board (wood and nails)
10. Stereogram (wood)

Some of the techniques used in the construction of models may be illustrated by a few examples which follow. However, a more complete treatment may be found in such discussions as those by Giebel, Fehr and Hildebrandt, and the Committee on Models and Visual Aids of the Association of Mathematics Teachers of New York City.[8]

[8] Giebel, K., *Anfertigung Mathematischer Modelle.* B. G. Teubner, Berlin, 1915. Fehr, H. F. and Hildebrandt, E. H. C., *The Construction and Use of Mathematical Models.* State Teachers College, Upper Montclair, N. J., 1938. Association of Mathematics Teachers of New York, *On Models and Visual Aids.* New York, 1939.

Model No. 1: Plane Geometry. For the proposition, if $a > b$ then $A > B$.

Materials. Cardboard—1 piece, white; 1 piece, white on one side and red on the other.

Construction. On a moderate-sized piece of white cardboard draw a scalene triangle with side CB greater than CA (see Fig 3).

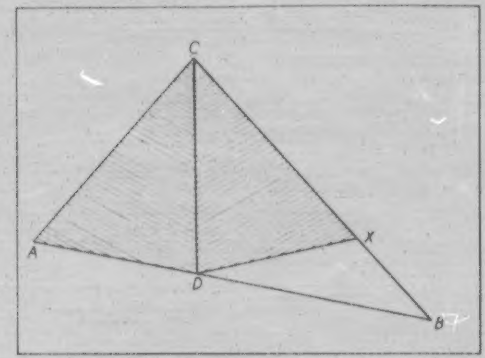


Figure 3

Bisect angle C and mark off CX equal to CA. Connect X with D, the intersection of the bisector with AB. Cut out of the red and white cardboard a figure congruent to $CADX$, and draw the line CD on the white side. Score along CD so that the quadrilateral hinges and folds back on the red side.

Fasten the triangle ACD in place, white side down, with either paste, staples, or both, leaving triangle CDX free to fold back and forth. When CDX is folded back over ACD, we have a white triangle ABC. When it is unfolded, we have the figure for one of the proofs of this theorem. The lines CD and DX should be accentuated with black ink.

Model No. 2: Spherical Trigonometry. To show the relationship: $\cos a = \cos b \cos c + \sin b \sin c \cos A$.

Materials. One piece of cardboard, white on both sides.

Construction. Lay out a sector of central angle 150° for a circle of 5" radius. Divide this sector into three sectors whose central angles are unequal; so that the sum of any two is greater than the third. Call the center O and the radii OA, OB, OC, and OA_1. The three central angles will be c, a, and b, in order. Extend the radii OB and OC (see Fig. 4). Draw the tangent to arc AB at A and extend it to meet OB at B'. Draw the tangent to arc A_1C at A_1 and extend it to meet OC in C'. Connect C' and B', and on this line as base construct a triangle with AC' and AB' as the other two sides.

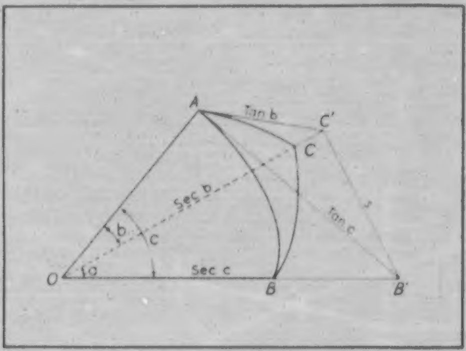


Figure 4

This triangle should be made to fall outside the sector; its vertex may be denoted as A_2. Cut away the figure bounded by the lines OA, AB', $B'A_2$, A_2C', $C'A_1$, A_1O.

Now draw the lines on the reverse side of the cardboard corresponding to those on the front. Score along the lines OB', $B'C'$, and OC' on the back and fold forward along these lines. The result will be a tetrahedron O—$A\ B'C'$ with a spherical triangle ABC drawn on its faces and a fourth face perpendicular to the edge OA.

The relationship is proved in the following manner:

In triangle $AB'C'$, by the law of cosines:

$$s^2 = \tan^2 b + \tan^2 c - 2 \tan b \tan c \cos A.$$

Similarly in triangle $OB'C'$:

$$s^2 = \sec^2 b + \sec^2 c - 2 \sec b \sec c \cos a.$$

Subtracting the second equation from the first and combining:

$$0 = 2 - 2 \sec b \sec c \cos a + 2 \tan b \tan c \cos A.$$

Multiplying by $\frac{1}{2} \cos c \cos b$ we get:

$$0 = \cos b \cos c - \cos a + \sin b \sin c \cos A.$$
$$\therefore \cos a = \cos b \cos c + \sin b \sin c \cos A.$$

Model No. 3: Solid Geometry. Cube with inscribed octahedron.

Materials. Red, white, and blue applicators, one dozen of each color.

Construction. On a piece of cardboard draw a square 4 inches on a side. Cut four blue applicators (hereafter called sticks) and miter them to coincide with the square. Drop some *airplane glue* or *collodion* on the corners and allow to dry. The glue will seem to disappear; in reality it flows around the sticks and makes a smooth joint. It is better to apply small quantities of the glue at intervals of a minute than to apply a large quantity at one time. Repeat this process on another piece of cardboard so that it will not be necessary to wait until the first dries. Allow the two squares to dry for about ten minutes.

Cut four blue sticks about one-eighth inch shorter than the side of the square. Remove the two squares from the cardboard, cutting them free if they stick, and prop them up so that they are perpendicular to the cardboard. Place two sticks on the cardboard between the two squares and glue them in place. Allow them to dry and glue the remaining two edges. The cube is now defined by its twelve edges.

Cut six white sticks the length of the diagonal of a face of the cube. Starting from any vertex, place one diagonal in each face. Place only six diagonals and make sure that there are three meeting at each of the vertices A, C, F, H (see Fig. 5). The six white sticks determine a tetrahedron inscribed in the cube. The other diagonals determine another tetrahedron. They might be added in another color if desired.

Mark the mid-points of the six white diagonals and call them a, b, c, d, e, f. With cutting pliers make a slight nick at each of these points. Cut the red sticks to fit between c and d, d and e, e and f, and f and c. If these are cut with pliers, the ends will be wedge-shaped and can be made to rest in the nicks mentioned above. Glue in place and allow to dry. Place the cube so that b is at the bottom; fit the pieces bc, bd, be, and bf and glue in place.

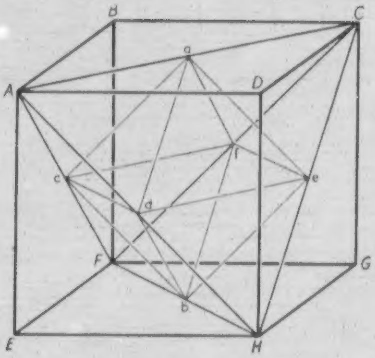


Figure 5

This operation is rather delicate and will require some patience. Repeat with the lines *ac*, *ad*, *ae*, and *af*, and the model is complete.

To make this model of glass, use six perfectly square pieces of glass, joined as shown in Figure 6. The tetrahedron can be folded from paper and its vertices glued to the glass before the sixth side is attached. The glass can be joined with any transparent cement or airplane glue. Black binding tape, such as that used for binding lantern slides, may also be used. The ordinary 2″ x 2″ glass lantern-slide covers are superb for a small model. The arrangement in Figure 6 shows two tiny cubes omitted in two

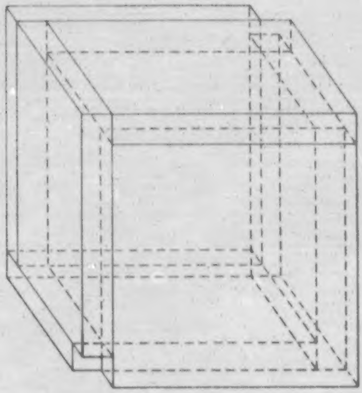


Figure 6

corners. These may be ignored because their size is only the thickness of the glass, or they may be filled with glue.

The following properties of the regular solids suggest other models of this type: the diagonals of the faces of a cube determine two inscribed tetrahedrons; the mid-points of the edges of a tetrahedron joined in succession form an octahedron; the mid-points of the faces of an octahedron joined in succession form a cube; the diagonals in the faces of a dodecahedron form five inscribed cubes; the body diagonals of a dodecahedron form a stellated icosahedron; and the body diagonals of an icosahedron form a stellated dodecahedron.

Model No. 4: Geometry of Space. Hyperbolic paraboloid.

Materials. Applicators and a piece of wire (optional).

Construction. Bend a piece of wire into the form of a rhombus and bend two adjacent sides out of the plane of the other two, thus forming a space quadrilateral. Mark off about ten equally spaced points on each of the sides of the quadrilateral. Lay an applicator between a point and its corresponding point on the opposite side and glue in place. Repeat for each of the marks on the two sides. The set of lines joining the two sides is one set of generators of the hyperbolic paraboloid. By repeating this operation with the remaining two sides, we get an overlapping set of reguli which also determine the surface. Figure 7 shows a surface made in this fashion.

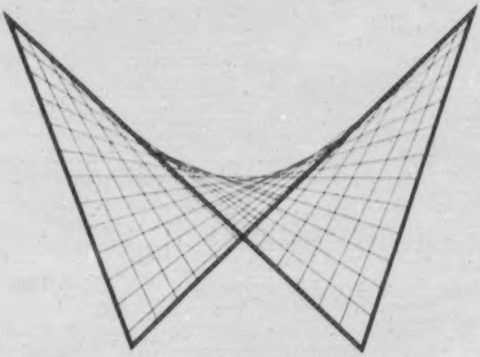


Figure 7

The joining of corresponding points of any two skew lines in space will yield such a surface. The non-adjacent sides of a tetrahedron, being skew lines, will answer the purpose and provide a very neat method of constructing such a surface. This construction, accomplished without the use of wire, is composed entirely of applicators.

Glue six applicators together to form a tetrahedron. Mark off ten equally spaced marks on each of two opposite edges. Nick the sticks at each of the marks with pliers and glue applicators between corresponding points as noted above. The result is very pleasing.

Still another way to make this model is to build a hinged skewed quadrilateral and string round elastics through screw eyes fastened in the wood sides of the quadrilateral. This will provide a variable model (see Fig. 2, model 8) which is very effective.

Model No. 5: Plane Geometry. Theorem board.

Materials. Plybord, 20″ x 20″; wire brads; elastic bands.

Construction. In the center of a piece of plywood paint a circle, the circumference of which is one-quarter inch wide, and with diameter 15 inches. The plywood need not be more than one-quarter inch in thickness. Drive small wire brads around the circle at intervals of about one-half inch. Outside the circle drill a few holes large enough to accommodate a golf tee. These tees are to be used for stringing tangents and secants to the circle.

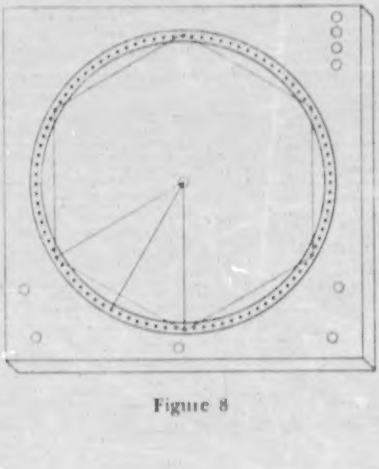


Figure 8

Drill a hole at the center of the circle and place a golf tee in it. With this device it is possible to set up the figures for all the propositions on circles and for many other theorems. The lines are formed by large elastic bands or round elastics (see Fig. 8).

Model No. 6: Solid Geometry. A collapsible spherical frame.

Materials. Steel strapping, brass eyelets.

Construction. Cut four strips of *strapping* about three-quarters of an inch longer than the circumference of the sphere desired. Mark a point one-half inch from the center of the strip and punch a hole to fit the eyelets available. From the center line mark off on either side a distance equal to half the circumference. Punch a hole at each of these marks and fasten with an eyelet. There will be a piece which overlaps. Punch a hole in the double thickness and fasten with an eyelet. Remove the first eyelet, providing two diametrically opposite holes which will be used to pivot two more circles. These are constructed in a similar fashion except that they are made slightly smaller so that they fit inside the first. To make these, punch a hole three-quarters of an inch from the end of a strip and coil it inside the completed circle, punched hole innermost. With a scriber scratch a mark on the other end of the strip to match the punched hole. Remove, punch a hole at the mark, and fasten together to form a second circle. Punch a hole adjacent to the eyelet in the double thickness and fasten again. The purpose of the two fasteners is to provide a rigid joint which will not twist out of the plane.

It is not necessary to mark the two diametrically opposite holes before forming. They can be punched when the circles are placed inside the first circle made, using the first holes as guides. Since the overlapping parts are thicker than the rest of the circle, it will not do to let them pile up in one place. They should be staggered, as in Figure 9. Thus only on the first circle will the diametrically opposite pivotal holes coincide with a joint in the strip.

Pivot three circles to form three meridians. Construct a fourth to fit outside the first. Punch two diametrically opposite holes in it. Place the first inside the fourth, and turn the fourth so that its holes are turned ninety degrees with respect to the pivotal points of the first. Punch through these holes, thus providing an equatorial axis. Fasten the first and fourth at this axis. The result is a sphere defined by four great circles, three of which are pivoted at a common pole; the fourth has its pole at right angles to the

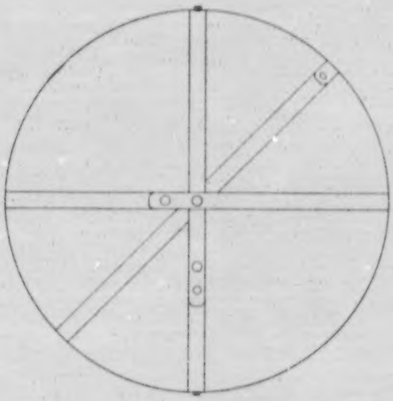


Figure 9

other. This model is collapsible and can be used advantageously in solid geometry, spherical trigonometry, navigation, or astronomy.

A successful teacher can also be a bit of a showman. There is nothing which impresses a student more than an exhibition of craftsmanship at work. It is ever a source of surprise to find that "teacher" can use his hands as well as his head. The success to be met in the use of these models depends more on careful preparation and proper selection than on any special skill. Materials used in class demonstrations should be within arm's length. A newspaper picked up deliberately and folded along a diagonal in a sure and confident manner is far more valuable in a demonstration than one put on after fumbling in a rear closet for a few minutes for a ready-made model, no matter how perfect and polished the latter may be. A razor blade in a handle, a paper punch, some brass paper fasteners, a supply of shirt cardboards, and strips of wood of various length—these are the "stock in trade" of the "teacher-modelmaker." It is advisable to rehearse each demonstration, or at least to have some practice in using the knife. Unforeseen complications can arise in the execution of some simple experiment, but these need not be serious; experience is the best teacher. Provision should, of course, be made to avoid marring the desk; a couple of sheets of cardboard should be placed over the top, or if more protection is desired, a piece of ¼″ fir plywood one foot square.

The value of models lies in their availability to the teacher and the class at the psychological moment. A model locked in the teacher's drawer or closet is of no use to anyone. Just as important as the development of a permanent collection is the necessity of providing adequate showcases or other facilities for housing it. Demonstration models should be so located that the teacher need only reach for them. This does not mean that all the models should be stored around the teacher's desk. Provision may be made to store those in current use within easy reach.

Certain types of models lend themselves readily to mounting on the bulletin board, the wall, or an open ledge around the room. Artistic distribution of posters and models about the room lends a mathematical atmosphere which will serve to stimulate student interest in the subject.[9]

We must not fail to mention the benefits derived by the student in making models. The completion of a model gives a feeling of satisfaction and accomplishment. The admiration shown by the rest of the class pays for [illegible] time spent in its construction. Often it is the weaker student who makes the best model. Hence, he should be encouraged, for it is he who most needs to "square himself" in the eyes of his fellow students. The use and display of a beautifully made student-model will result in increased appreciation of the importance of good workmanship, neatness, and accuracy, and will stimulate others to improve existing models or to plan new ones. A good collection of models is not assembled in a day or a month or a year, but it is amazing how the collection grows and the dividends it pays.[10]

[9] Cook, Mary Ruth, "Stimulating Interest in Mathematics by Creating a Mathematical Atmosphere." *The Mathematics Teacher*, Vol 24, pp. 248-254, April, 1931.

[10] Mossman, Edith L., "A Math Room That Speaks for Itself." *School Science and Mathematics*, Vol. 35, April, 1933.

BALSA AND CLAY AS TEACHING AIDS IN HIGH SCHOOL MATHEMATICS

Harriet B. Herbert

TWO MATERIALS, not in general use have been found most helpful in clarifying geometry problems. They are the light balsa sticks used in making model airplanes, and commercial modeling clay used with balsa sticks or with the "Fiddlesticks" or "Pick-up Sticks" which may be purchased in toy stores.

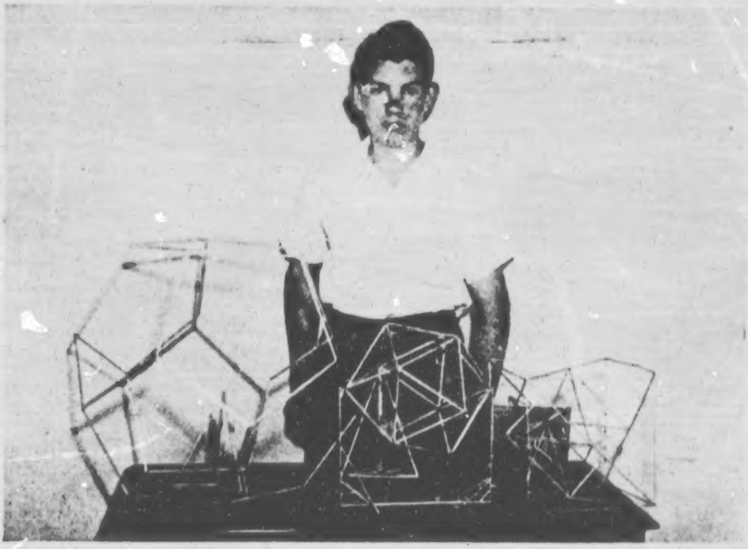


Figure 1. Allie Phifer and His Regular Solids

Most boys delight in making neat constructions with balsa. They spend hours on a regular dodecahedron or icosahedron, mitering the corners with the aid of a razor blade and patiently applying a bit of airplane cement at every joint. They will also read most carefully the statement of a difficult problem, build a model according to its specifications, and gain new understanding for themselves and for their classmates.

My most unusual experiment with a class using balsa has been building the tesseract, $(a + b)^4$, an original problem in the fourth dimension.

One day, in listing some twenty suggestions for individual work in my plane geometry class I included on the mimeographed variety program: "Make a balsa model of $(a + b)^3$ to demonstrate the binomial theorem of algebra," the first steps of which are shown in Figure 2. In presenting this particular suggestion, I reminded the class that, when they learned that $(a + b)^2 = a^2 + 2ab + b^2$, they had probably drawn a square $a + b$ on a side, and easily divided it into a^2, a square with side a; $2ab$, rectangles with sides a and b; and b^2, a square with side b. I sketched this square on the board. Anyone who cared to do so was to construct such a square, $(a + b)^2$, of sticks; and on that square, divided into a^2, $2ab$, and b^2, to build a cube to show that $(a + b)^3$ is composed of a^3, $3a^2b$, $3ab^2$. and b^3.

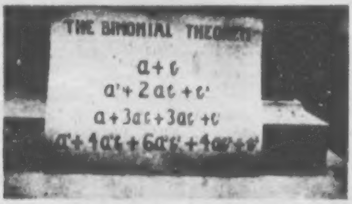

THE BINOMIAL THEOREM
$a + b$
$a^2 + 2ab + b^2$
$a^3 + 3a^2b + 3ab^2 + b^3$
$a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

Figure 2. Expansion of the Binomial Theorem

Next day a girl brought in the first cube, a good one except for the fact that she had put it together with pins where a boy would have used glue. Then a number of well-made cubes followed. Among these was a large one built on a fourteen-inch edge, with the short a ends of the sticks painted brown and the long b ends green. The boy who made this one hinted that he liked such work with sticks and glue better than writing a paper on Thales, and asked for another job like it. I told him I could think of nothing better than to try to carry out this construction to the fourth dimension.

The fourth dimension was a new idea to all of my classes. The figure I wished to introduce to them was the tesseract, the name given to the extension of a cube to fourth dimension. Just as a second dimension is added to a line to form a square and a third to a square to form a cube, a fourth is added to a cube to represent a tesseract.

Because we had only three dimensions in which to represent a tesseract, we were in the position of a person wishing to draw a cube on paper. Perspective had to be used. The two accepted ways of representing a tesseract follow exactly the two ways of drawing a cube. The slightly sidewise view of a cube consists of a foreground square and a background square joined by shortened oblique parallel lines. The direct view of a cube shows a square within a square, as if one were looking directly into a box or down a well. Similarly, the first tesseract in Figure 3 has a foreground cube and a background cube joined by parallel oblique lines slightly shorter than the other lines to give the illusion of perspective, while the second figure is a cube within a cube.

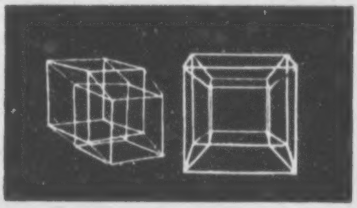


Figure 3. The Two Views of the Single Tesseract

Having learned what a tesseract would look like when represented in three-dimensional space, the boys added the fourth dimension to their large cube painted brown and green. The foundation cube being already divided into cubes and rectangular prisms, it remained for us to divide all the newly added lines into $(a + b)$ lengths. To do this completely was really complicated in practice, but simple enough in theory. Soon, every square on the whole model had become $a^2 + 2ab + b^2$; every cube was $a^3 + 3a^2b + 3ab^2 + b^3$; and it was yet to be shown that the complete tesseract was $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$. The direct view of $(a + b)^4$, at right in Figure 4, was made later.

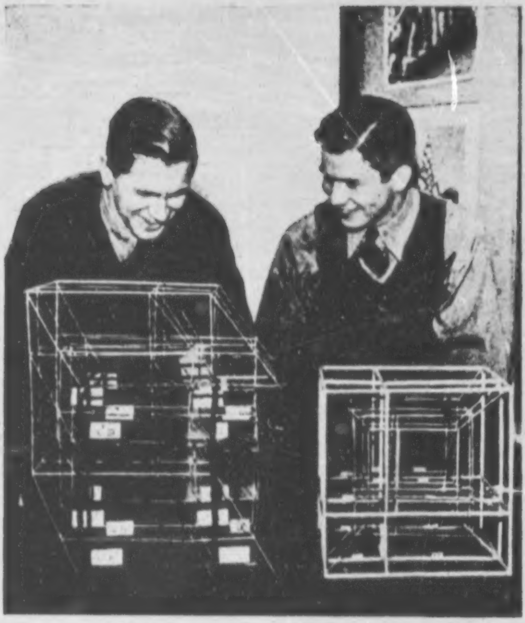


Figure 4. The Tesseract $(a + b)^4$

We easily identified a^4, a cube with side a being its foundation cube, with an added dimension of length a. Some of our art students made diagrams showing the other forms that would be present in $(a + b)^4$. Figure 5 shows the effect of adding a dimension to a^2b or to ab^2. Since, at every vertex of a^2b, three lines meet, two short a's and one long b, the resulting figure, after

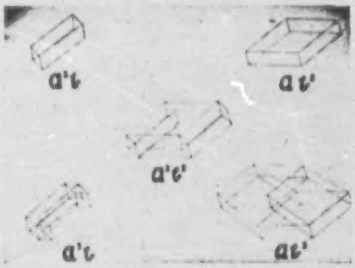


Figure 5. Adding the Fourth Dimension

another dimension is added, will have at each vertex an added a or b, and will thus be either a^3b or a^2b^2. There are, similarly, two possibilities for ab^2, which may become either a^2b^2 or ab^3. We were able to identify and label, not only the two tesseracts, a^4 and b^4, but also the fourteen additional hyperprisms corresponding to the other terms of the expansion, namely, four a^3b's, six a^2b^2's, and four ab^3's.

An interesting feature of this type of geometry—adding dimensions—is that still more algebra is involved. Seeing that in a square two lines meet in a point, in a cube three, and in a tesseract four, we found the number of sticks needed to make each model, using the theory of combinations. Also, by counting vertices, edges, faces, and so on, the students discovered that these

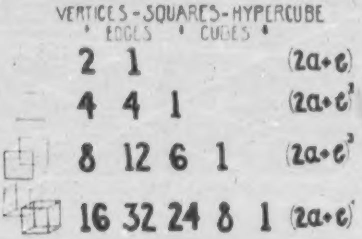


Figure 6. Counting the Points and Lines

numbers are the binomial coefficients. Figure 6 demonstrates the known theorem for the series, line, square, cube, and tesseract. In counting the eight cubes of the simple tesseract (see Figure 3), we find six of them distorted, just as four of the six squares in the drawing of a cube are distorted. In Figure 3 the six distorted cubes of the first figure appear as parallelepipeds with rectangular bases, while in the second figure they are truncated pyramids with rectangular bases.

A high school boy easily carried the simple tesseract on to the fifth dimension (see Figure 7). He also built a simplex, a figure founded on the equilateral triangle and regular tetrahedron, and extended it to the sixth dimension (see Figure 8). These two constructions, either of which could have been carried to still higher dimensions, led us into more geometry to be interpreted algebraically. The models shown as Figures 7 and 8 are entirely original. Figure 8 is of colored sticks. It began with a single stick of one color. Two, of a new color, were added to make the triangle. Three, of another color, meeting at an outside point, formed the tetrahedron. It will be noted that the number of sticks used to add each dimension is equal to the number of the new dimension.

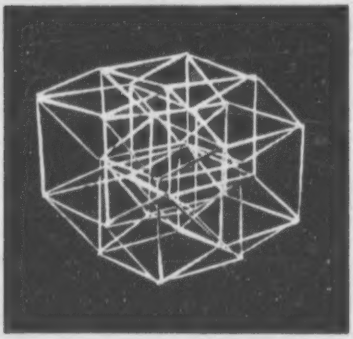


Figure 7. The Fifth Dimensional Analogue of a Cube

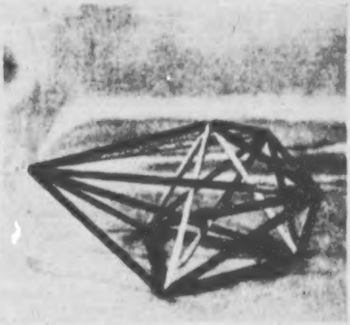


Figure 8. The Simplex of the Sixth Dimension

While the mathematics classes were interested in the building and study of fourth dimensional models including the regular polyhedroids, the art instructor had her classes make original designs suggested by our models of the regular polyhedroids. Some

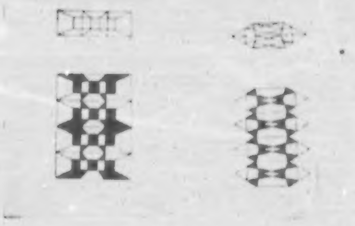


Figure 9. Designs Suggested by Regular Polyhedroids

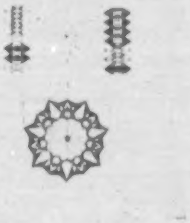


Figure 10. Designs from Polyhedroids Elaborated

of their designs are shown in Figures 9 and 10. The students plan further to design wrought-iron gates and, possibly, stained glass windows by modifying these designs and developing others.

Our use of clay, the other material we have brought into the geometry class, involves no stretching of geometry beyond the bounds set for that subject by Euclid; in fact, it demands no intricate mathematics whatever. On the contrary, to use a clay foundation on which to build a figure of "fiddlesticks" or "pick-up sticks" provides a simple and quick way to get a figure before the class in solving problems for which a drawing seems insufficient.

We knead clay and flatten it into a hose box. In the smooth clay we can actually bisect angles or construct perpendiculars. We use the clay box, however, chiefly as a foundation in which to stand sticks forming a pyramid, a parallelepiped, or other figure. The "pick-up sticks" purchased for a game of the same name are very good for this purpose. When sticks are brought together at a point, an extra bit of the clay serves to hold them there. After the problem is solved, the sticks are removed, any drawing in the basal clay is erased with the pressure of a finger, and the clay box is ready for the next problem. There is no trouble with leftovers as in the case of cardboard and string models, too good to be destroyed before the eyes of their creators, but not worth handing down to posterity.

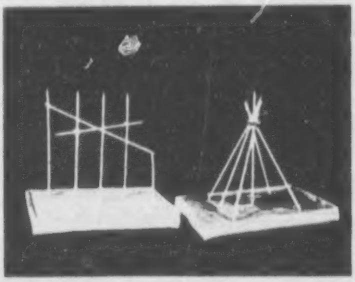


Figure 11. Clay Boxes with Fiddlesticks

The first model in Figure 11 should give a clue to the solution of the theorem, *The perpendicular from the intersection of the diagonals of a parallelogram upon a plane that does not cut* *the parallelogram is equal to one-fourth of the sum of the perpendiculars from the vertices of the parallelogram upon the plane.*

The second model in Figure 11 would help an amateur in solid geometry to see the right triangles involved in the problem: *The altitude of a regular quadrangular pyramid is six inches. Each side of the base is four inches long. Find the lateral area.*

By the time we get to spherical geometry, the class is usually more independent and less in need of props for its thinking. However, some find it hard to visualize spherical triangles. For these the use of clay may again be helpful. The clay is removed from the box and smoothed over a globular bowl. It is very convincing to make the drawing in the clay, with the help of compasses that have been bent to reach a quadrant's distance and to demonstrate that *If one spherical triangle is the polar of a second, then the second is also the polar of the first.*

In these days, hundreds of boys and girls are studying mathematics not because they love it or are particularly good at it, but because of the urgency of the demand for many to become expert in some of its applications. Therefore, it seems more necessary than ever that teachers find ways of making mathematics more attractive for them. We may find inspiration almost equal to that in teaching a brilliant student, in having these lesser ones say, "Gee, it's interesting when you really see it!"

HISTORICAL MATERIAL ON MODELS AND OTHER TEACHING AIDS IN MATHEMATICS

Lao Genevra Simons

IN ALL WORKS on solid geometry and related subjects, a figure in a plane is made to represent a figure in three-dimensional space. One stumbling block at the beginning of solid geometry has always been the making of the solid emerge from the plane figure, the seeing of a solid in a plane. This was especially true before laws of perspective were followed in the drawings in a book on the subject. This stumbling block may not exist, but there may be the problem of drawing a figure in a plane that shall represent in two dimensions a solid in three. There are students and teachers as well who can easily see a solid in the air, as it were, and can follow the demonstration of a proposition connected with it, but who cannot make a drawing of it. Both the difficulties suggested tend to :nake the first lessons in solid geometry critical ones.

Since the solid must be seen either in the imagination or in the actual object, the obvious means of smoothing the path is the building up of models. This is a legitimate aid at the outset, although it should be discarded as the course proceeds. As the student advances in his studies, he should grow away from dependence on the physical and grow into the purely mathematical expression of all geometrical figures.

The recognition of this need of models is found in the first English translation of Euclid's *Elements.* This work is entitled: *The Elements of Geometrie of the most auncient philosopher Euclide of Megara faithfully (now first) translated into the Engliske toung,* by H. Billingsley, Citizen of London. London, 1570.

An excellent presentation of this work by Professor Walter F. Shenton of American University appeared in *The American Mathematical Monthly* for December, 1928, under the title "The First English Euclid." The article included the following statement: "The second volume contains the tenth to fifteenth books of Euclid, with a sixteenth book added by Flussas. The wood-cuts illustrating the solid figures are most beautiful, but one of the most interesting features of the eleventh book is that many of the figures are made of paper and so pasted in the book that they

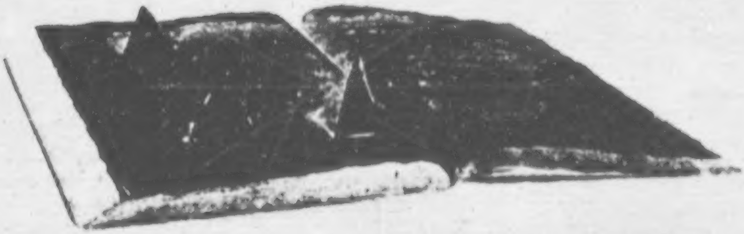


Figure 1. From Billingsley (1570). *Courtesy of Professor Shenton.*

may be opened up to make actual models of the space figures." Then the article gives figures; one "shows the page of definitions of pyramids, and illustrates with actual models, pyramids with triangular, quadrilateral, and pentagonal bases" (see Figure 1); another "shows similar figures to illustrate the proofs of such theorems as 'From a point geuen on high, to draw vnto a ground plaine superficies a perpendicular right line.' "

Early in the eighteenth century, another English translation of Euclid employed this same device but to a lesser degree. This work was entitled: *Euclid's Elements of Geometry from the Latin translation of Commandine. To which is added a treatise of the nature and arithmetic of logarithms; likewise another of the elements of plane and spherical trigonometry.* By Doctor John Keil, F. R. S. Now done into English. . . . By Mr. Samuel Cunn. London, 1723.

The first Latin edition of John Keil's *Elements* was printed in 1715. He justifies the appearance of another edition in print on the ground that attempts have been made to supersede Euclid's work because of the detection of numerous faults. He maintains among other things that "his Demonstrations [are] elegant, perspicuous and concise, carrying with them such Evidence, and so much Strength of Reason . . ." To the *Elements,* Keil adds some trigonometry with a reason that sounds very modern: "Farther, for the Use of those who are desirous to apply the Elements of Geometry to Uses in Life, we have added a Compendium of Plain and Spherical Trigonometry, by means whereof Geometrical Magnitudes are measured and their Dimensions expressed in Numbers."

The popularity of the Keil work is shown by the fact that it was translated into English and that the English translation ran through at least five editions, the fifth appearing in 1745. Mr. Cunn claims that he has corrected some errors, particularly in the solution of certain cases of "Oblique Sphericks." The geometrical part of the work follows the usual order of Euclid, Books 1 to 6, 11, and 12, but in a much abbreviated form. The diagrams are found on one sheet at intervals throughout the book. But it is the section on trigonometry which offers special interest from the standpoint of models. At the end of the work, there is a sheet from which figures can be erected to be used in connection with certain propositions. There are just three of these figures, but each one serves to illustrate several theorems.

The upper figure is a separate circle pasted onto the sheet. To it at a diameter is pasted a circle, half of which can be lifted up. The appropriate lines are drawn and letters attached for the several theorems, seven in number. The middle model (see Figure 2) consists of a circle (foreshortened in the figure) pasted to the sheet. Along two radii *OA* and *OH* are attached two plane right-angled triangles *OAI* and *OHG* which can be made to stand upright for use in two propositions. These are: "Prop. 23. In spherical triangles *BAC, BHE,* right-angled at *A* and *H,* if the same acute angle *B* be at the base *BA,* or *BH,* then the sines of the hypotenuses shall be proportional to the sines of the perpendicular arcs; Prop. 24. The same things being supposed, *AQ, HK,* the sines of the bases, are proportional to *IA, GH,* the tangents of the perpendicular arcs." The lower model serves for two theorems. Its foundation is a circle pasted to the sheet. To this is attached a sector pasted along a radius but not on a diameter. This sector can be made to stand up.

In the same year with the appearance of Cunn's translation of Keil's *Euclid,* Edmund Stone published *The construction and principal uses of the mathematical instruments. Translated from the French of M. Bion.* London, 1723. This work contains plates

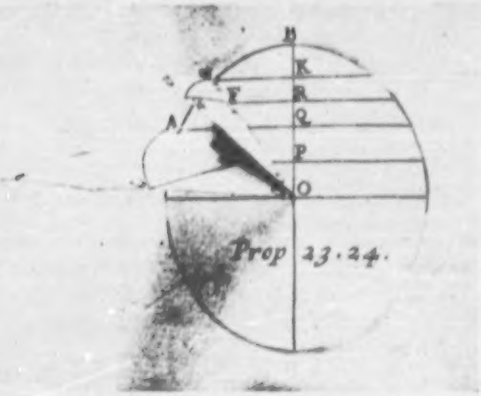


Figure 2. "Stand-up" Model from Cunn's Translation of Keil's *Euclid.*

with the usual diagrams for making the regular solids and a drawing of the appropriate solid itself to the left of each diagram. While this work purports to be a means of making mathematics practical and useful in the affairs of life, it does give some suggestions for making the theorems in solid geometry clear.

Further evidence that these "stand-up" figures were finding favor appears in the work: *Geometry made easy; or, a new and methodical explanation of the elements of Geometry: to which is added a new . . . method of exhibiting in miniature the various kinds of solids . . . by schemes cut out of paste-board.* By John Lodge Cowley. London, [1752].

A copy of this book is in the British Museum, as noted in catalogue of the same. One has not yet been located in this country. The contents of the work are outlined on page 3 of another work by this author as follows: "The work here alluded to was compleated in the year 1752, and contains a very easy and concise commentary on the first 6, 11, 12 and 15 books of Euclid [books 14 and 15 were added later than the time of Euclid himself]; some material propositions of Archimedes, concerning the cylinder, cone, and sphere; the principles of algebra, with its application in solving geometrical problems; and an introduction to conic sections; together with a method of forming, in miniature, solids and their sections, by schemes cut out of paste-board, which render it useful to those young students for whom it was designed."

The same author published: *An appendix to Euclid's Elements. In seven books. Containing forty-two copper plates. In which the doctrine of solids, delivered in the eleventh, twelfth, and fifteenth books of Euclid, is illustrated and rendered easy by new-invented schemes cut out of paste-board.* London, 1758.

At the outset, the author refers to his earlier work and then adds: "I come now, according to the promise which I made in the preface to that work, to lay before the public a new performance the chief scope of which is to produce mechanical representations of solids and their sections" and much more along this line.

A second edition of *An appendix to the Elements of Euclid* appeared in [176–?] with a slight variation in the title. "Containing forty-two copper-plates" has been replaced by "forty-two moveable schemes." Cowley justifies his second printing in his preface when he says: "The approbation shown to the first edition of this work, and the many applications made for it while out of print, are motives that have encouraged me not only to issue this second edition, but also to make sundry additions and improvements to it. . . . but as these things would augment the present work too much, and that those who have the first edition hereof may have an opportunity of obtaining these additions without detriment to their former purchase, they are reserved for a second or supplemental volume to what is herein contained." Such a supplemental volume has not been located in any library or catalogue and so there seems to be no extant volume to show that it ever appeared in print.

The "moveable scheme" employed may be described by taking an illustration from the book. To make the Exoctoedron or Canted Cube (see Figure 3), the outline of the entire figure,

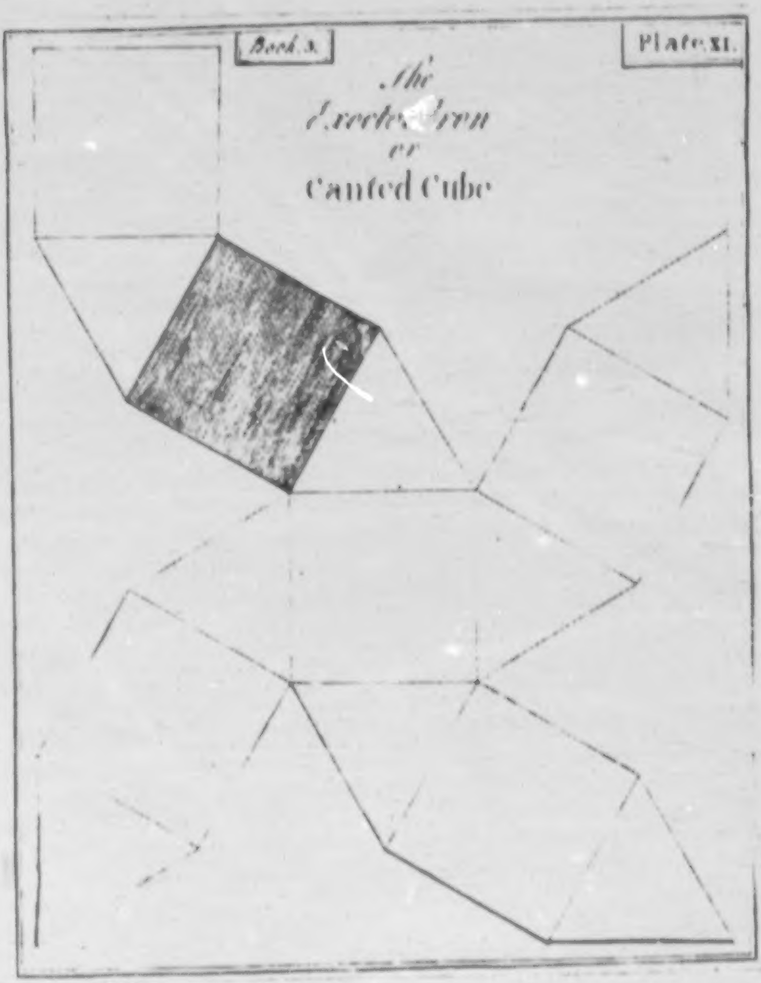

Book 3.
Plate XI.
The
Exoctaedron
or
Canted Cube

Figure 3. From Cowley's *Appendix*. Outline of Model to Be Raised

except for two sides of the shaded square, is cut out. The several lines (here twelve in number) which cross the diagram are so pressed that the[illegible]n be easily folded. Thus the six equal squares and eight equal equilateral triangles can be placed in such a position that the solid with the shaded square as base can be formed (see Figure 4). In the words of the author, ". . . nothing remains to hinder the reader from a clear sight of the true form of these bodies but the little trouble of raising up the figure, and folding its several parts around that particular one which is distinguished from the others by being shaded." This is the familiar means for making a solid. Here it is already at hand in the book. The disadvantage of the scheme appears at once. Only propositions which relate to solids whose entire surface can be flattened into one continuous figure can be represented. Such propositions as have been described in the Keil book are not possible by this scheme.

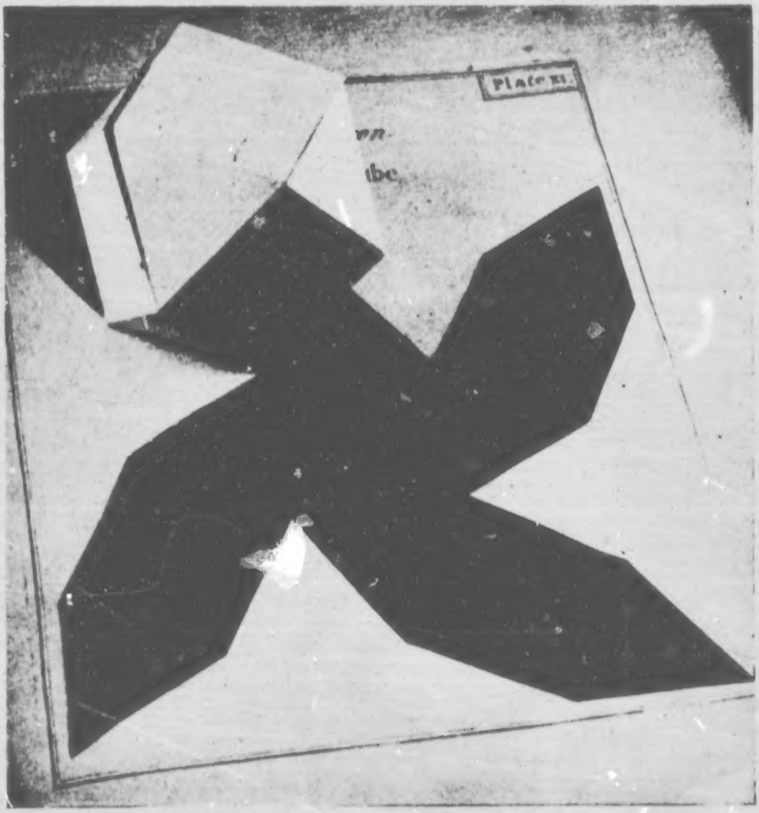


Figure 4. Model of Figure 3 in Upright Position

Nevertheless, Cowley does present a number of interesting solids and propositions. Book I exhibits the five regular solids; Book II, one regular solid inscribed in another, to be used with Euclid XV, 1-5; Book III, five of the irregular solids (these are five out of thirteen of the semi-regular polyhedra which were investigated by Archimedes. Kepler showed how these figures can be obtained), also solid rhombus, solid rhomboids, dodecarhombus; Book IV, various sorts of prisms; Book V, various kinds of pyramids, and their frustums; Book VI, the treatment here is especially commended by the author in these words: "In the sixth, the propositions of Euclid's eleventh and twelfth books are clearly explained, and so illustrated, as renders them perfectly easy to be comprehended, without perplexing the mind to conceive lines as drawn through such and such points, as raised up and meeting together at such and such points above the plane, having only an imaginary existence in the mind that conceives them; for the reader has here each solid really formed, and is at the same time furnished with an ocular view of the sections, justly made and laid open to his sight: wherefore, by the peculiar contrivance here made use of, there is no longer any difficulty in perceiving in the most clear and convincing manner . . ."; Book VII, the cone and the several sections which result in an isosceles triangle, a circle, an ellipse, a parabola, and a hyperbola. (See Figure 5 for the very nice device here employed.)

William Jones, "Mathematical Instrument Maker, accidentally purchased the plates" of the Cowley *Appendix* and published: *An illustration and mensuration of Solid Geometry; in seven books: containing forty-two moveable copper-plate schemes for forming the various kinds of solids, and their sections; by which the doctrine of solids in general, and those in the eleventh, twelfth, and fifteenth books of Euclid are elucidated, and rendered more easy to learners than by any work hitherto published.* By the late John Lodge Cowley. The third edition. London, 1787. In this volume, Jones claims to have revised, corrected, and augmented the Cowley work, but the changes do not affect the models in it.

The three editions of the Cowley *Appendix* indicate that it met a real need of the time. Jones might have been writing a preface for a modern educational work when he introduced the third edi-

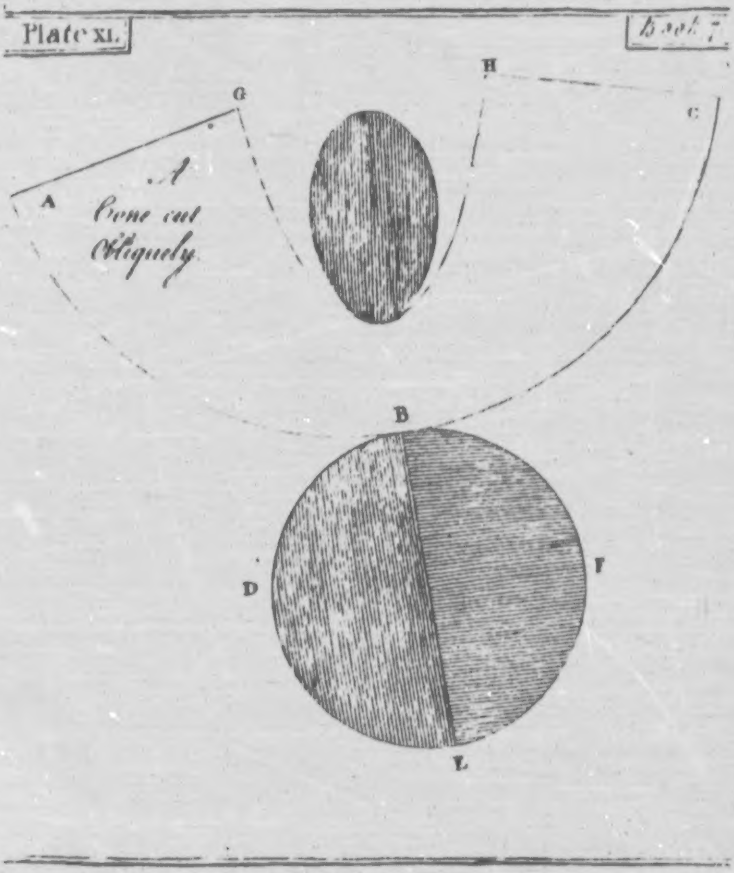


Figure 5. From Cowley's *Appendix*. Outline of Model to Be Raised to Show an Ellipse as a Section of a Right Circular Cone

tion by saying, "Probably in no branch of the mathematics does the student meet with more embarrassment than in the study of the geometry of solids; for most of the authors who have treated thereon have adopted the usual method of copying the abstruse and perplexed linear schemes of the ancients, which for the greater part are so inconsistently delineated in perspective as to render it impossible, even to Euclid himself, to divine the bodies intended to be represented by them. It was, undoubtedly, from a consideration of this kind, that Mr. Cowley was excited to any attempt of exhibiting the various kinds of solids treated of Euclid, and other geometricians, by moveable and folding paper schemes or figures, by which a person of the most slender capacity should at a small, or no expense of time have a clear and rational idea of the several solids, and thereby be able to investigate their properties with more perspicuity and precision."

A note in "A second list of books and pamphlets in the Library of the Mathematical Association," London, 1929, states that ". . . F. C. M. Marie reproduced 24 of Cowley's figures, with acknowledgment, to form a volume *Géométrie Stéréographique*, 1835."

Before the appearance of the edition by William Jones, another author adopted the device of "stand-up" figures, and entitled his publication: *A Royal Road to Geometry; or an easy and familiar introduction to the mathematics.* By Thomas Malton. London, 1774. An excellent and complete article, by L. Leland Locke, on this book appeared in *Scripta Mathematica* for March, 1941. Mr. Locke calls it "A Book Old Enough to Be Considered New" and points out a number of respects in which it anticipates the so-called "new" approaches in geometry of recent years.

The interest in Malton's work for the subject of models lies in its use of the scheme of Billingsley although on greatly improved lines (for "Malton is the author of a large treatise on perspective in which the same type of figure is utilized.") It is an improvement on Cowley, for we have here only the base of the figure pasted to the page instead of a part of the figure. Hence "several pieces may be used which would overlap if they had to be cut from the page." Malton may have been influenced by Keil as well as by Billingsley. In his preface, he refers to this author when he says, "Dr. Keil in his Preface to his Translation of Commandine and also Mr. Cunn seems to think it an unpardonable fault in Tacquet to omit the demonstrations of the 5th book. . . ."

No real study has been made of extant mathematical models in wood and brass. In *Early Science in Oxford Part II Mathematics* (R. T. Gunther, Oxford University Press. London, 1922, p. 36) the author lists the mathematical models at Oxford. This list includes the following: "Models for demonstrating Propositions in Euclid. Before 1697. Formerly in Savile collection, now missing [known from a manuscript of 1697]"; "Two 4-inch Demonstration Spheres, bisected on the ecliptic. c.1650. These two small spheres of beechwood are all that remains of the elaborate instrumental outfit of the Savilian Professors which was kept in the *Cista Mathematica* [the great Mathematical Instrument Box of the University]"; "Three Demonstration Spheres. c.1700 . . . Marked in ink to illustrate propositions in spherical triangles. Two of the spheres are supported in turned wooden pedestal cups." (See Figure 6.)

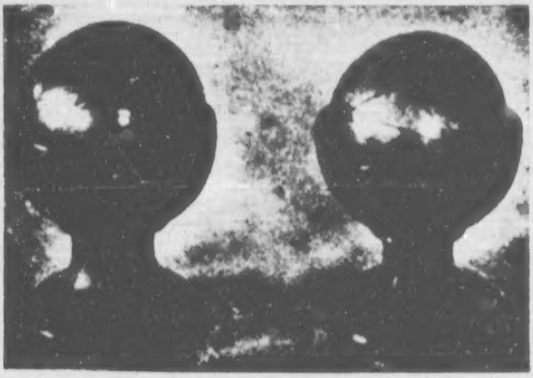
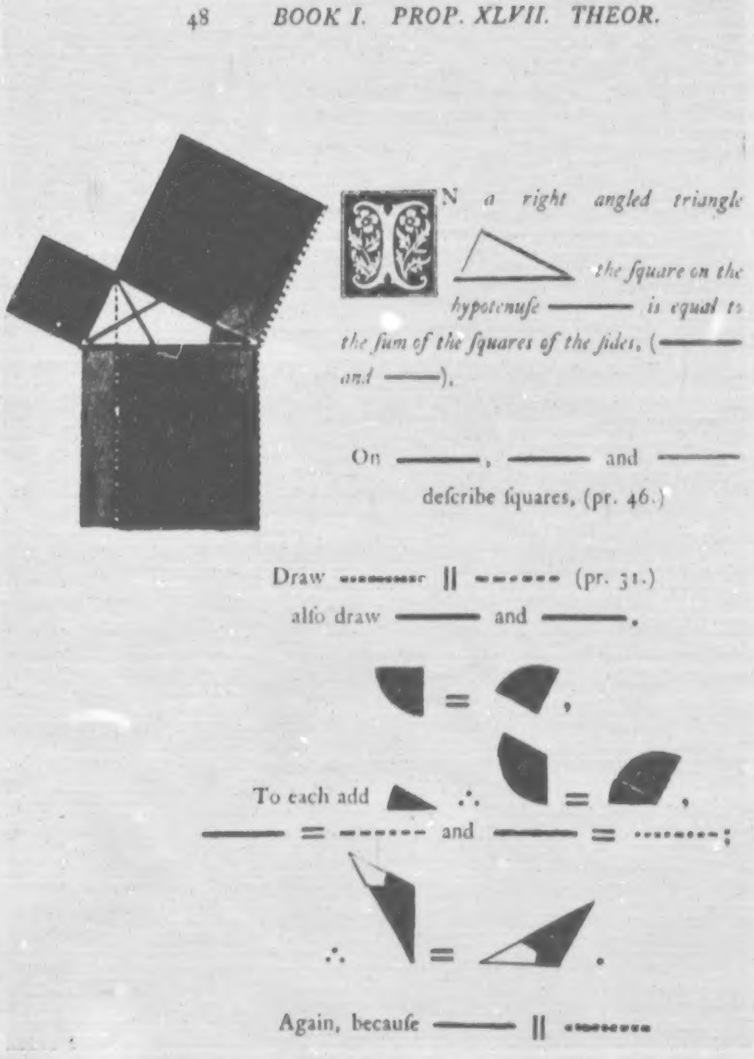


Figure 6. From Oxford Collection of Models. Permission of Oxford University Press

There are spheres in use today which resemble the last-named solids. These are small slated globes (spherical blackboards) which rest in a standard of iron and which are to be used by the student in spherical geometry and trigonometry.

This brief historical sketch may serve to show that the use of models as teaching aids in mathematics is not new. The periodic revival of any aids and projects seems to indicate that methods move in cycles and this is no criticism of the individual methods.

Further lines in which mechanical aids appear are color, paper-cutting and folding, the Chinese puzzle, and, doubtless, others. The first mathematics text to use color, and that consistently, was an English publication entitled: *The First Six Books of the Ele-*



48 *BOOK I. PROP. XLVII. THEOR.*

IN *a right angled triangle* *the ſquare on the hypotenuſe* ——— *is equal to the ſum of the ſquares of the ſides,* (——— *and* ———).

On ———, ——— and ——— deſcribe ſquares, (pr. 46.)

Draw ········ || ------ (pr. 31.)
alſo draw ——— and ———.

= ,

To each add ∴ = ,
——— = ------ and ——— = ········;

∴ = .

Again, becauſe ——— || ········

Figure 7. Pythagorean Theorem from Byrne's *Euclid* in Color

ments of Euclid in which coloured diagrams and symbols are used instead of letters for the greater ease of learners. By Oliver Byrne. Surveyor of Her Majesty's settlements in the Falkland Islands. . . .

London, 1847. The name of the printer, William Pickering, indicates that it was regarded as a work of importance, but no evidence can be adduced to prove that it influenced later works. In it color runs riot and it must be regarded as a freak presentation. At the same time, it probably was the first work in print to suggest that colored chalk or pencil can be really helpful in complicated geometric diagrams.

The line of treatment will be indicated by reference to the diagram for the Pythagorean Theorem (see Figure 7). The colors are employed as follows: In the right triangle, shorter side yellow, square on it black; longer side blue, square red; hypotenuse red, square in two rectangles, to left blue, to right yellow; continuous lines which cross triangle black, dotted line red; angles at lower right vertex of triangle, clockwise, blue, black, yellow. Only four colors are used throughout and the color is solid. The proof proceeds by use of colored diagrams. Here, in words to name the colors, blue angle equals yellow angle. To each add black angle and so forth.

The only books that approach this one are *New Plane Geometry* by Webster Wells (Boston, 1908) and *New Solid Geometry* (same author and date). The former contains three plates in which the sides of triangles are outlined in color; the latter contains two plates in which planes are shown in solid color. Plate V is for the proposition *The volume of any parallelepiped is equal to the product of its base and altitude.*

It is not the intent of this brief survey to recommend any of these aids for present-day use. An examination of the books themselves, as opportunity presents itself, is recommended instead. There can be no more enjoyable and uplifting experience than to see at first hand the worth-while efforts of an alert teacher like Mr. Cowley to make the subject of mathematics a vital thing to his students. This is the ambition of a teacher worthy of his calling.

Note: The author wishes to acknowledge the courtesy of Miss Anne S. Pratt, reference librarian of Yale University Library, in sending a copy of the first edition of Cowley's *Appendix;* also, of the Plimpton Library, Columbia University, in providing a second edition and illustrations from the same; finally, of the late Mr. L. Leland Locke, who loaned his copy of the third edition, one of the fine old volumes in the library of this collector of good books.

THE CONSTRUCTION OF PLAITED CRYSTAL MODELS

Summary by Virgil S. Mallory

AN INTERESTING method of constructing models of various polyhedra is explained in a little book now long out of print. The unique method used will undoubtedly have sufficient interest for teachers of mathematics to justify the summary given here.

The title page describes the book as "A System for the Construction of Plaited Crystal Models on the Type of an Ordinary Plait; Exemplified by the Forms Belonging to the Six Axial Systems in Crystallography; by John Gorham, M.R.C.S. Eng., &c., Tonbridge, Kent; E. and F. N. Spon, London, 1888; IV + 28 with 56 Plates 24 pp. Adv."

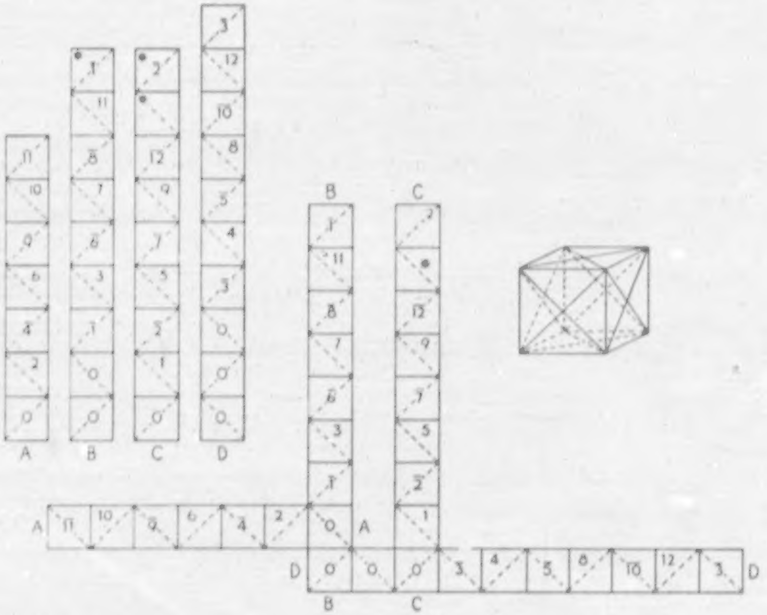
The author states in the preface: "It is now some forty years since I had the honour of demonstrating before the Royal Society in London *A System for the Construction of Crystal Models Projected on Plane Surfaces.* These figures folded into the required form, and subsided into a level at pleasure—they were easily moulded into shape by bringing their edges into apposition with the fingers, and were as easily transferred from place to place when flattened in a portfolio—they constituted, in short, an extension of the plan used in modern mathematical treatises for extemporising models of the five regular or Platonic solids. . . . Each [of the six crystalline] systems consists of a skeleton model of three or four rods of wood, wire, or glass; these rods are called axes, round which the forms can be symmetrically built up. Upon these axes it is proposed, in the first place, to find the faces of the required model by direct measurement (or recourse may be had to Spherical Trigonometry, as the case may be), and in the next place, to build them up into a model by a process which it is believed has not been hitherto attempted. It consists in taking an ordinary plait of three or four rushes, defining its intersections in numerical order, and thus eliminating the *type* on which every model is constructed. By strictly adhering to the type it was found, moreover, that those solids which were confessedly irregular and most difficult to understand—those, for instance, belonging to the *doubly oblique system*—were made with the same facility as the cube itself. . . . On a careful examination of one of these crystals, its faces appeared to be arranged in narrow strips, which could be traced round the form in four different directions, and seemed to cut each other in their course as if they intersected. It became difficult not to realize the practicability of using strips of paper—of crossing them just as these zones appeared to be crossed in a real crystal, and of intertwining them as in a plait. Four narrow strips of paper were taken accordingly, each being composed of similar rhombs placed together at their opposite edges, after crossing and recrossing repeatedly, a rough model of the form was eventually obtained. . . . It became evident that the definite arrangement of the parts in a plait could be at once utilised by finding the numerical order in which its intersections occurred. This formed a clue to the whole."

Since the primitive forms belonging to the six axial systems in crystallography may all be constructed from that for the cube, the author emphasizes that form, though plates and descriptions are given for other forms, including crystals showing cleavage, striation, and pyramidal hollows in the faces, as in crystals of bismuth and of potassium bromide and iodide.

The cut (p. 268) shows the four plaits used in forming the model of a cube. The numbers show the order of plaiting, the face numbered I coming under that marked 1, etc. Faces marked * are to give stability to the model, while those marked 0 are base planes. Dotted lines indicate folding. The resulting cube has each face composed of four isosceles right triangles.

Modifications to produce other interesting forms in crystallography are produced by effecting changes in the angles of the isosceles triangles. When the vertex angle is 90°, the cube results; when it is 83°, a pyramid of four lateral faces is raised on each face as in some varieties of quartz; while, if it is reduced to 70° 31′ 44″, the resulting model is that of the rhombic dodecahedron like garnet crystals.

In forming the model of a rhombic dodecahedron, the plaits shown in the figure here are modified by replacing each square by a rhombus, the shorter diagonal being parallel to the dotted (folded) lines. If the faces are now numbered in the same way, the pattern will plait into a firm model.



In a similar manner the author shows the modifications to the cubic model necessary to produce the other five axial systems. In every case the type of axes tells (1) the form and angles of the faces and (2) the method of locating the sides. From the cubic plaits the mode of adjusting the faces to form a plaited model is obtained.

A high school mathematics club will find the construction of these plaited crystalline forms an interesting and instructive project, particularly if the construction of the forms is accompanied by an elementary study of crystal axial systems.

COLOR IN GEOMETRY

L. Leland Locke

ONE OF THE first books on plane geometry to use colors in identifying parts of the Euclidean figures was *The First Six Books of the Elements of Euclid in Which Coloured Diagrams and Symbols Are Used Instead of Letters for the Greater Ease of Learners, by Oliver Byrne, London 1847.*

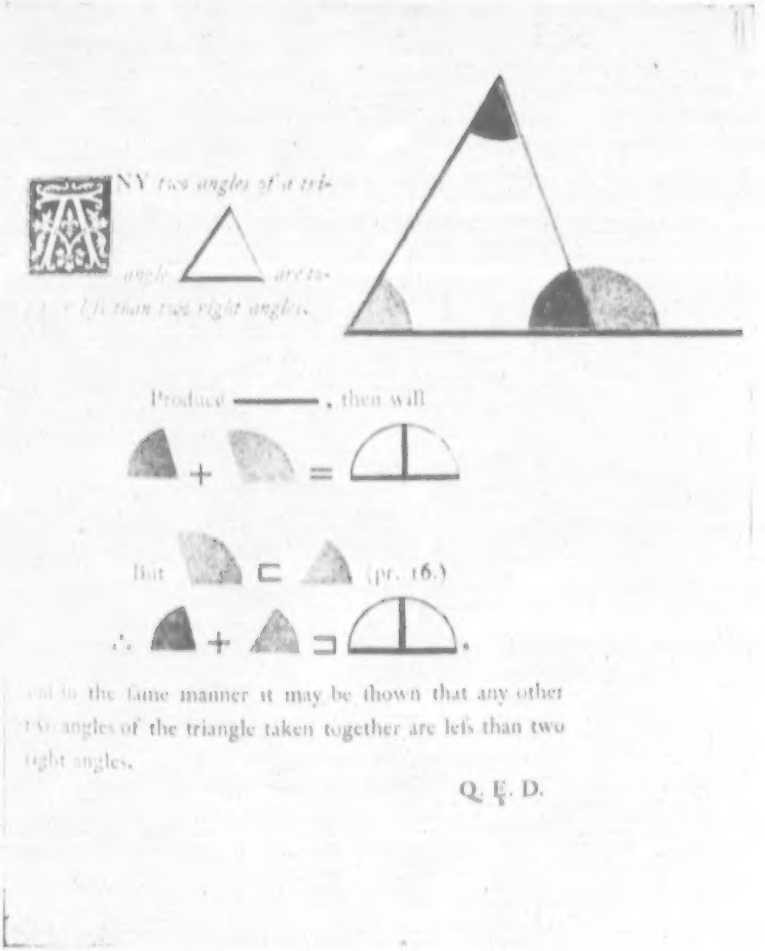
The extracts from the Introduction which follow may serve to illustrate the principles set forth by Byrne in his book.

"The plan here adopted forcibly appeals to the eye, the most sensitive and the most comprehensive of our external organs, and its pre-eminence to imprint it subject on the mind is supported by the incontrovertible maxim expressed in the well known words of Horace:

> A feebler impress through the ear is made,
> Than what is by the faithful eye conveyed."

"The letters annexed to points, lines, or other parts of a diagram are in fact but arbitrary names, and represent them in the demonstration; instead of these, the parts being differently coloured, are made to name themselves, for their forms in corresponding colours represent them in the demonstration."

"In oral demonstrations we gain with colours this important advantage, the eye and the ear can be addressed at the same moment, so that for teaching geometry, and other linear arts and sciences, in classes, the system is the best ever proposed. Besides the superior simplicity, this system is likewise conspicuous for concentration, and wholly excludes the injurious though prevalent practice of allowing the student to commit the demonstration to memory; until reason and fact, and proof only make impressions on the understanding. Again, if we mention the colour of the parts referred to, as in saying, the red angle, the blue line, or lines, the part or parts thus named will be immediately seen by all the



ANY *two angles of a tri-*

angle △ *are to-*

gether leſs than two right angles.

Produce ——, then will

+ =

But ⊏ (pr. 16.)

∴ + ⊐ .

and in the ſame manner it may be ſhown that any other two angles of the triangle taken together are leſs than two right angles.

Q. E. D.

Figure 1

class at the same instant, not so if we say the angle ABC, the triangle PFQ, the figure EGK, and so on, for the letters must be traced one by one before the students arrange in their minds the particular magnitude referred to, which often occasions confusion and error as well as loss of time.

> Sounds which address the ear are lost and die
> In one short hour, but these which strike the eye
> Live long upon the mind, the faithful sight
> Engraves the knowledge with a beam of light."

Figure 1 is an illustration of a typical page showing the proof of a theorem of plane geometry. In the triangle shown, the right side and its opposite angle are red; the base and the vertical angle are black; and the exterior angle is shown in yellow. The contrast of these colors is partly emphasized by the reproduced page shown in the figure.

The demonstration reads:

> "The red angle plus the yellow angle equals two right angles.
> But the yellow angle is greater than the blue angle (pr. 16)
> Therefore, the red angle plus the blue angle is less than two right angles. Q.E.D."

Note that Byrne omits the customary particularization of the theorem in the form "Given" and "To prove."

Both Mr. Morris Cohen of the mathematics department of the Boys Technical High School, Brooklyn, New York, and the writer have found that a class in plane geometry will welcome with zest an occasional excursion into color, using colored chalk on the blackboard and colored pencils on paper. Mr. Cohen has devised a variant of the plan for printing purposes to avoid the expense of colored plates. For the form of Figure 2, the designations of

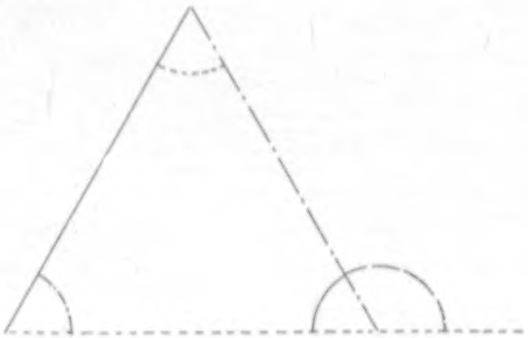


Figure 2

lines and angles might be black, white, dotted, dashed, dot-dash, double dot-dash, or other variations. Figure 3 shows a simplified form, adapted to blackboard and paper use.

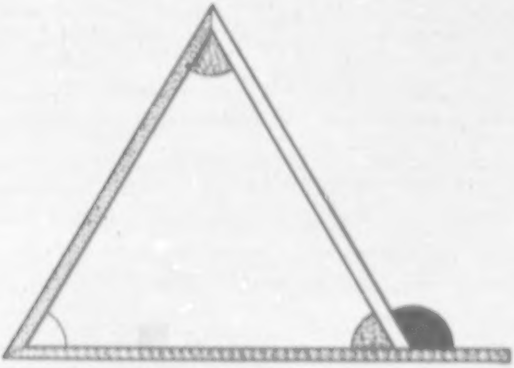


Figure 3

The Byrne volume is not only a beautiful addition to the antiquarian's collection, but it has some pedagogical value. A recent book, "Introduction to Geometry," by A. W. Siddons and K. S. Snell, Cambridge University Press, 1939, contains on one of its pages four figures for proofs of the Pythagorean Theorem in color, another figure for parallel lines, and two others on congruent triangles. So the basic pedagogy of Byrne, with its combined appeal to the ear and eye, may, under present-day printing methods, soon be used to a greater extent in our plane geometry texts.

AN EARLY WORK ON MECHANICAL DEVICES FOR DRAWING THE CONIC SECTIONS

Phillip S. Jones

ALTHOUGH mechanical devices for drawing curves date back at least to Nicomedes (270 B.C.), who used one such for drawing the conchoid which he designed to trisect angles,[1] and although devices for drawing conic sections were first heard of not later than Proclus (A.D. 410-485), who discussed an ellipsograph,[2] nevertheless the book *De Organica Conicarum Sectionum in Plano Descriptione Tractatus* (A Treatise on Devices for Drawing the Conic Sections) by Franciscus van Schooten, printed at the Elzevir Press in Leyden, Holland, in 1646, is of much interest both in itself and in relation to the history of mathematics and of mechanical devices in particular. (See Figure 1.)

The first chapter is of some historical importance because of its description of a mechanical device for drawing a straight line by means of a link and a triangular ruler. The existence of this earlier work is little known and, as a result, A. Peaucellier is usually given the credit of being the first to construct a straight-line linkage with his inversor of 1864. In van Schooten's device shown in Figure 2, $AB = BC = BD$, with the result that as vertex D of the rigid triangle BCD slid along line AE the other vertex, C, of the triangle would trace the dotted line. For proof van Schooten shows that angle $CAD = \frac{1}{2}$ angle CBD. He notes that if CBD is a straight line, $CA \perp$ to AE. Peaucellier's device, differing from this, is a true linkage with no sliding parts and does not involve a previously given straight line.

This book is of further interest because it treats all the conic sections in a unified fashion, lists a variety of mechanical devices for describing each, and gives several different solutions to many of the construction problems that it sets up. For instance, in Chapter VIII the problem of constructing an ellipse, given the

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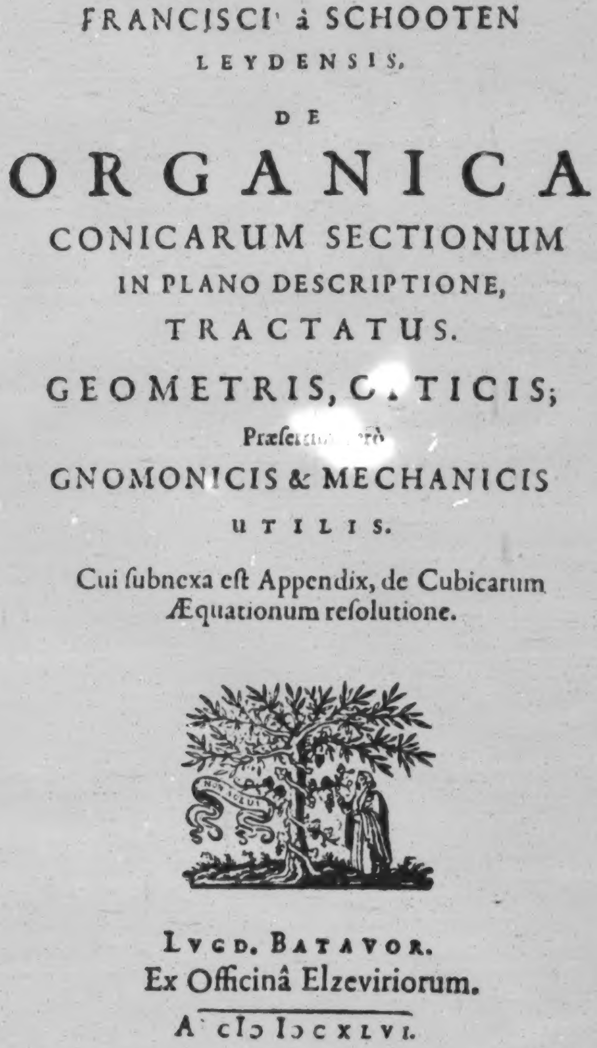
[1] Footnotes 1 to 4 refer to references at the end of this article.



FRANCISCI à SCHOOTEN
LEYDENSIS,
DE
ORGANICA
CONICARUM SECTIONUM
IN PLANO DESCRIPTIONE,
TRACTATUS.
GEOMETRIS, OPTICIS;
Præſertim verò
GNOMONICIS & MECHANICIS
UTILIS.
Cui ſubnexa eſt Appendix, de Cubicarum
Æquationum reſolutione.

LUGD. BATAVOR.
Ex Officinâ Elzeviriorum.
Aº CIↃ IↃ CXLVI.

Figure 1

foci and vertices, is treated. One solution given for this problem is the familiar pin and string construction which is based on the fact that the sum of the focal radii is constant. In the following

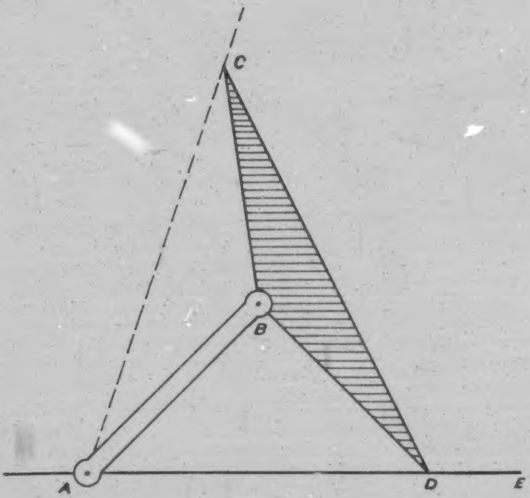


Figure 2

chapter the same problem for the hyperbola is treated by a similar series of devices, one of which, as shown in Figure 3, is a pin, ruler, and string construction (the string is fastened at *N* and *C*) based on the fact that for an hyperbola the difference of the focal radii is constant. A second solution to this problem for the ellipse was obtained by using the four-bar linkage shown in Figure 4. After

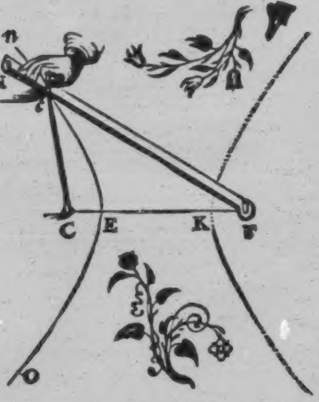


Figure 3

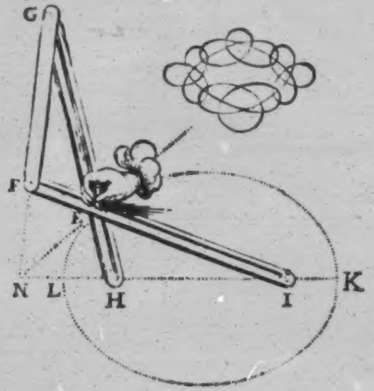


Figure 4

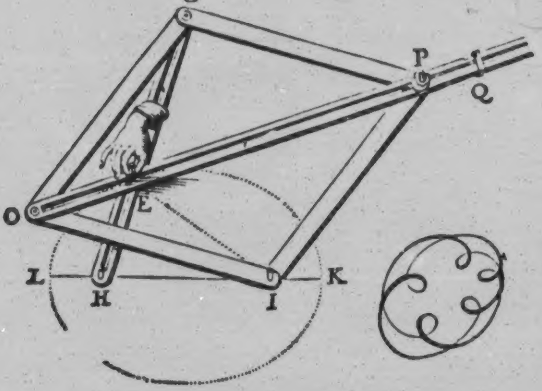


Figure 5

discussing this solution, van Schooten added a scholium showing how this same linkage could be used to determine the tangent *NE*.

Other problems solved mechanically are these: to construct an ellipse given the axes, given any pair of conjugate diameters, given

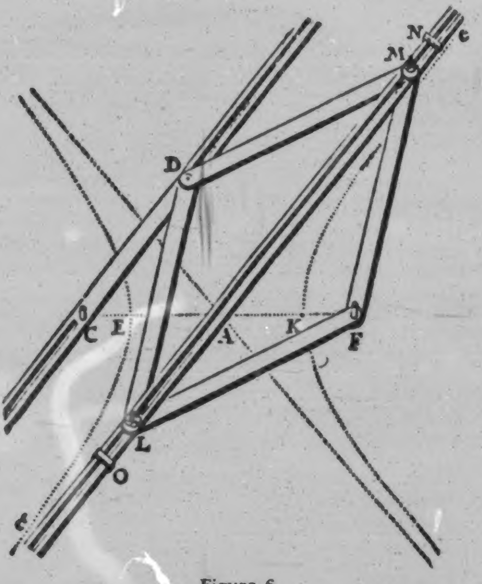


Figure 6

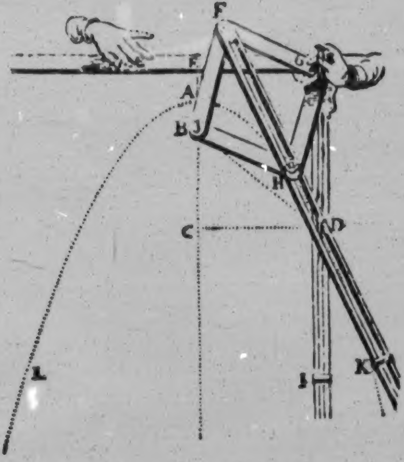


Figure 7

a latus rectum and the transverse axis; to construct an hyperbola given its axes, given any pair of conjugate diameters, given a latus rectum and transverse axis, or given a point and the asymptotes; to draw a parabola, given the axis, vertex, and latus rectum, or given the focus and vertex. Several theorems on the areas of parabolic segments are included at the end.

Figures 5, 6, 7 show what is essentially the same apparatus used to describe in turn an ellipse, an hyperbola, and a parabola, further stressing the family relationship. An additional construction for a parabola is shown in Figure 8.

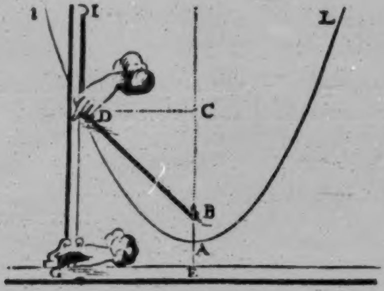


Figure 8

This book was one of the earliest by an author whose works and references span one of the most interesting periods in the history of mathematics. His father, Frans van Schooten, also a mathematician, in 1627 published one of the earliest trigonometric tables. The sources cited by van Schooten in this work are chiefly Euclid and Appolonius, with a few references to that forerunner of the calculus, the Method of Indivisibles of Cavalieri. The treatment of the problems in this book is non-analytic; nevertheless, van Schooten shows, through references in the introduction, familiarity with the works of Descartes, Fermat, and Roberval, contemporaries who were working with the concepts which were soon to lead to the analytic geometry and calculus. That van Schooten was also familiar with analytic methods is evidenced by his Latin translation (1649) of Descartes' *La Géométrie* with the notes of Florimond de Beaune. The rapid spread of the method of Descartes is in large part due to this translation into the universal language of scholarship of the day and the added explanations accompanying the translation of the sometimes obscure original text. Van Schooten himself applied analytic geometry to the solution of many problems in his chief work *Exercitationes Mathematicae* (1657)[3]. He also published a paper by Christian Huygens on probability and in the second edition of the Descartes *Geometria* papers by John Hudde on the solution of equations and maxima and minima. Van Schooten recommended the use of coordinates in three dimensional space and published treatises on perspective, plane trigonometry, and surveying.[4]

A final look at the title page, shown in Figure 1, of the work under discussion here reveals a few items of interest. Van Schooten advertises the conic sections as of use in geometry, optics, gnomonics (the theory of the sun dial), and mechanics. These claims he amplifies somewhat in his preface, telling of the reflecting or focal properties of the conics as studied in optics and of their use in the architecture of vaulted roofs and bridges. The appendix on the solution of the cubic equation is missing from the copy in the author's possession. The printer's device is one of several used by the Elzevirs. Finally, it may be of interest to note that in the date are used the earlier form of Roman numerals with cIↄ for 1000 and Iↄ for 500.

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2. DYCK, WALTHER. *Katalog Mathematischer und Mathematich-Physikalischer Modelle, Apparate, und Instrumente.* Dr. C. Wolf & Sohn, München, 1892. p. 58. See also p. 68 for a discussion of *De Organica.*
3. CAJORI, F. *History of Mathematics.* Macmillan Co., New York, 1919, p. 180.
4. *Bibliotheca Chemico-Mathematica.* Henry Sotheran and Co., 1921. Vol. I, p. 52; Vol. II, p. 737.

GEOMETRY WITH AN OPAQUE PROJECTOR

J. W. Colliton

For more than ten years I have been using an opaque projector in the teaching of my classes in plane geometry. The two classes, numbering thirty-five to forty each, are composed of students who have shown above-average ability in their elementary algebra. Their reactions to the use of the projector show how valuable other teachers may find this means of instruction.

This is the plan. I have drawn on 7″ x 10″ cards the figures for the textbook theorems and originals that I expect the class to study. These are thrown on a screen, at the front of a slightly darkened room, by means of an opaque projector. This practice saves the time required in having students draw the figures on the board and also permits a study of many figures and originals not found in the textbook in use.

On some cards the theorem is stated, and the students are asked to apply the hypothesis and conclusion to the lettered figure shown on the screen and then find a way to prove the theorem. In some cases, various members of the class will find several different methods of proof for one figure. On other cards the applied hypothesis

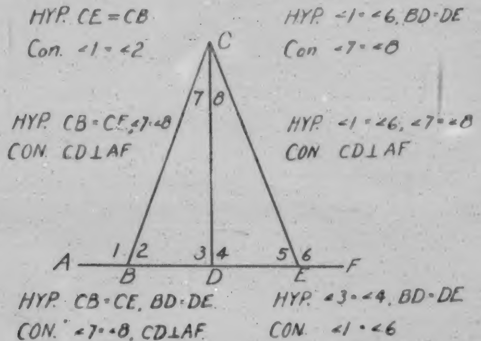
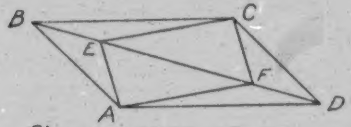


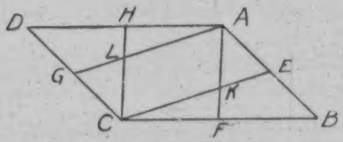
Figure 1



Given. ▱ ABCD, BE = DF.

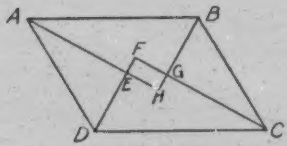
Prove AECF is a ▱.
If AECF is a ▱ and BE = DF is ABCD a ▱

Figure 2


D
H
A
L
G
E
K
C
F
B

Given: ▱ ABCD, & E, F, G, H the midpoints of AB, BC, CD & DA resp
Prove: AKCL is a ▱
Is AFCH a ▱? Is it a ▱

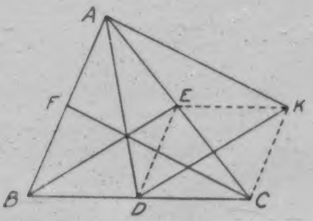
Figure 3



Given: ▱ ABCD, AH, BH, CF and DF bisectors of ∠s A, B, C and D.
Prove: EFGH is a rectangle.

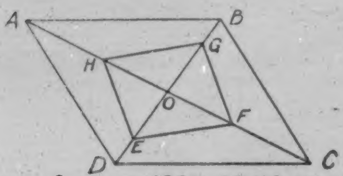
Figure 4

Given. △ABC, medians AD, BE & CF
DK = BE DK ∥ BE



Prove. KCFA is a ▱

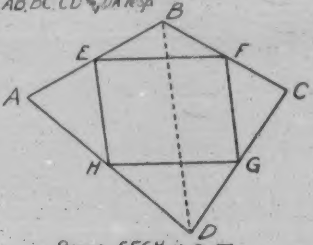
Figure 5



Given: ▱ ABCD, BG = DE, AH = CF
Prove: EFGH is a ▱
Suppose E, F, G & H are midpoints of OD, OC, OB & OA respectively?

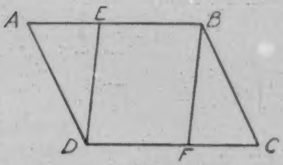
Figure 6

Given: Quad. ABCD, E, F, G, H midpoints of AB, BC, CD & DA resp



Prove. EFGH is a ▱

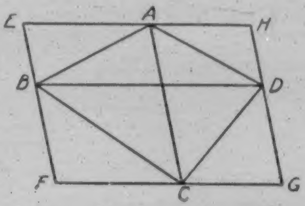
Figure 7



Given. ▱ ABCD, AE = CF

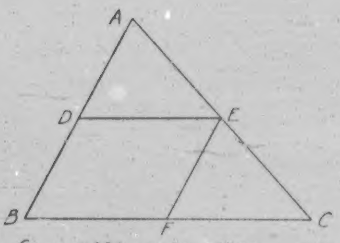
Prove. DEBF is a ▱

Figure 8



Given. Quad ABCD, EH ∥ BD, FG ∥ BD
EF ∥ AC, HG ∥ AC
Prove EFGH is a ▱. Area EFGH = 2 Area ABCD

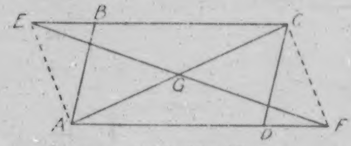
Figure 9



Given $\triangle ABC$, $AD = DB$, $DE \parallel BC$

Prove $BDEF$ is a ▱

Figure 10



Given ▱$ABCD$, $BE = DF$

Prove $EG = GF$

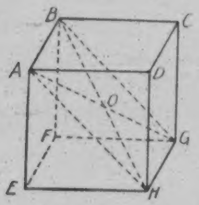
Figure 11

and conclusion are given; on still others only the hypothesis is given, and the class is asked to discover what conclusions, if any, can be drawn and proved. On some cards the figure is given, together with several different hypotheses and conclusions (Fig. 1) which may or may not be provable with the information that the class possesses at that point in its study of geometry.

Many figures may be used to give the class experience in the use of a theorem or a group of theorems. For instance, I have over forty cards with figures requiring the proof that a certain quadrilateral is a parallelogram, a rectangle, a rhombus, or a square. Some of these are shown in Figures 2 to 10. Others use the properties of these figures to prove lines equal or unequal (Fig. 11).

The simpler ideas of solid geometry can be easily introduced at appropriate times (Figs. 12 and 13).

Figure 14 shows how several numerical examples can be made from the same figure. In triangle *ABC*, *CD* bisects angle *C*. The



Given: Cube AG

Prove $AG = BH$

Figure 12

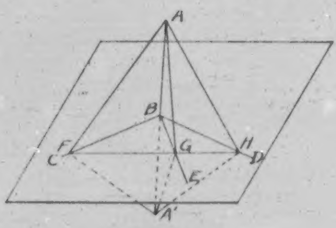
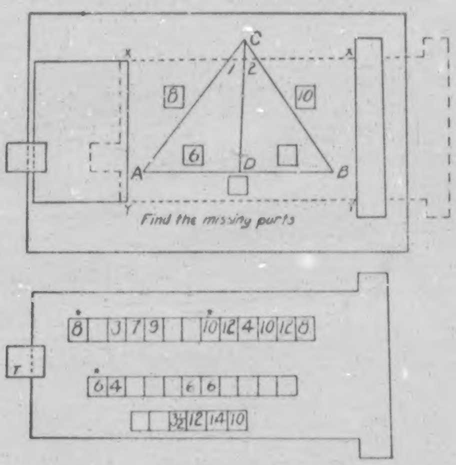


Figure 13

card is cut along the lines *XY* and holes are cut through it where the squares are shown. On another card shown in the lower figure, numbers are placed so that some show through the holes when



Lines at *XY* are slits through which lower card is slipped. *T* is a gummed linen tab to pull card to the left.

• These are the numbers showing through the holes in the upper card.

Figure 14

this card is slipped behind the upper card and through the slits *XY*. When the numbers for the two blank squares are found, the lower card is pulled slightly to the right so that a new set of numbers appears at some of the holes; the values are computed for those blanks, and the process is repeated for another problem.

Similar cards can be constructed for various figures showing the relationship of angles and arcs in circles; products of lines from a point and intersecting a circle; the right triangle with altitude drawn to the hypotenuse, and the relationship of the seven line segments in it; figures for areas of plane figures, and many others. The numbers involved should be relatively small but the results need not be whole numbers. Examples requiring the use of large numbers and extended computation should be given only for out-of-class work.

When the same theorem, such as the Pythagorean Theorem, can be proved by using any of a number of figures, many can be in-

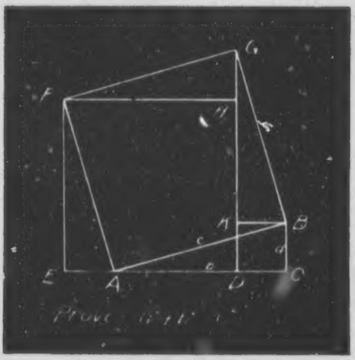

G
F
H
K
B
c
a
d
E
A
b
D
C
Prove

Figure 15

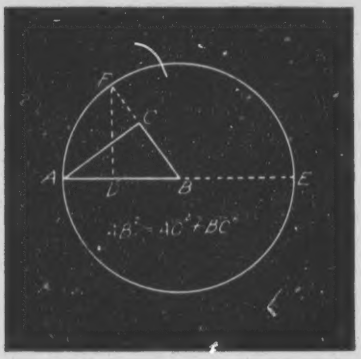

F
C
A
D
B
E
$AB^2 = AC^2 + BC^2$

Figure 16

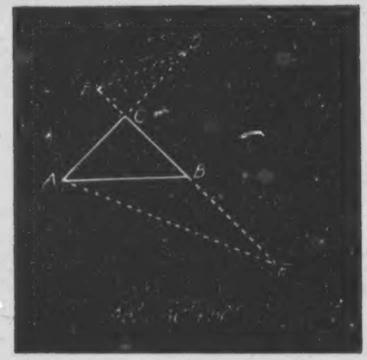


Figure 17

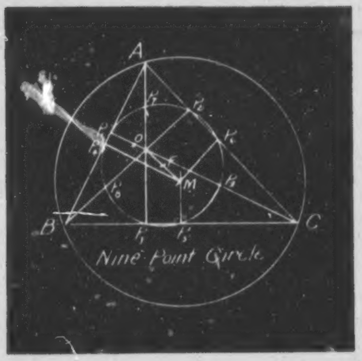

A
O
M
B
C
Nine-Point Circle

Figure 18

vestigated by a class if the figures are on cards which have only to be placed in the projector. I have about twenty different figures from which this theorem can be proved, but I do not use all of them every year. The number used depends on the interest the class shows in investigating the various proofs. See Figures 15 to 17.

When time permits, one can go a little beyond the usual theorems found in the textbook and bring to the attention of the class the Nine-Point Circle, the Theorem of Pappus, Ceva's Theorem, Menelaus's Theorem, and others. See Figures 18 to 22.

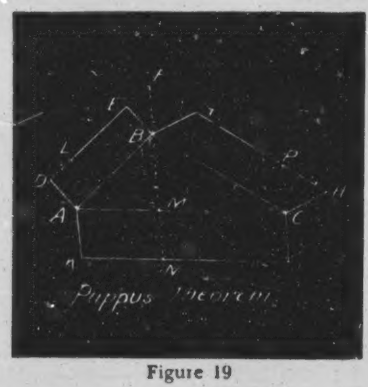

F
E
B
L
P
D
H
A
M
C
N
Pappus Theorem

Figure 19

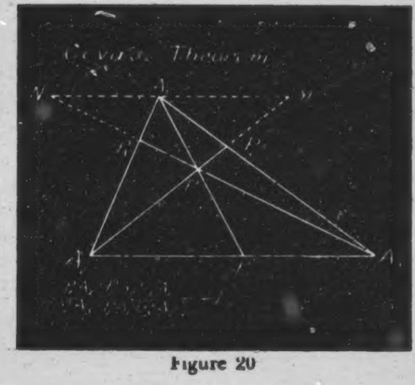

Ceva's Theorem
N
M
P
A
A
A
= 1

Figure 20

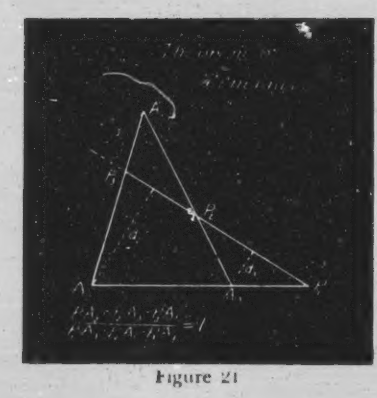


Figure 21

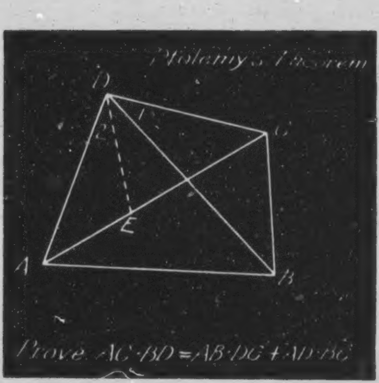

Ptolemy's Theorem
D
1
2
C
E
A
B
Prove: AC·BD = AB·DC + AD·BC

Figure 22

A
m
n
o
F
D
G
H
A
E
B

Figure 23

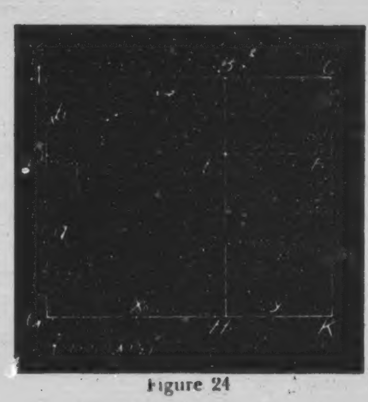


Figure 24

Constructions are studied by placing a large number of figures before the class with all construction lines shown (Fig. 23) and having the class discuss the plan for the construction by a study of the figure. In some cases the same figure is shown constructed by various methods on different cards. Algebra and geometry are related by such figures as 24-27.

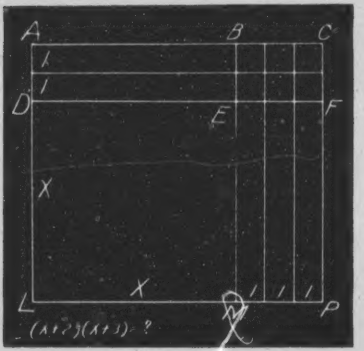


Figure 25

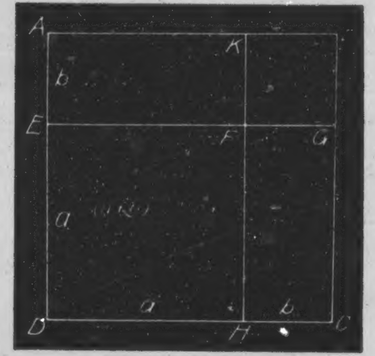


Figure 26

Skill in the use of the tools of construction cannot be gained in this way, but the ideas here gained can be applied to other constructions which the students should be required to do out of class. Those who have skill in construction or who have original

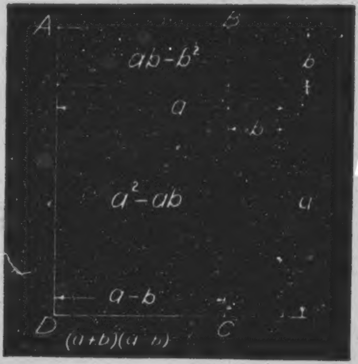


Figure 27

ideas are encouraged to make designs on cards suitable for projection. Some pupil-made designs are shown in Figures 28 to 32. Some pupils color their designs and some make tile or linoleum patterns.

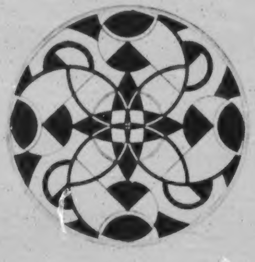


Figure 28

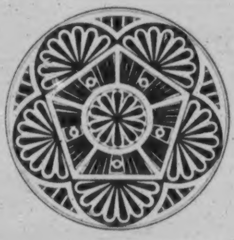


Figure 29

The history of mathematics is brought to the attention of the classes by means of pictures of great mathematicians mounted on cards. Thumbnail sketches of their lives and contributions are written up in a convenient notebook. These biographies are also used as special topics and add to the cultural value of the subject.

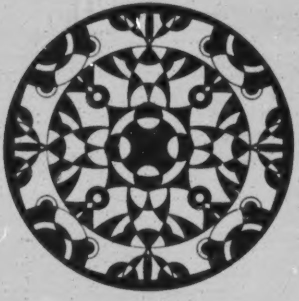


Figure 30

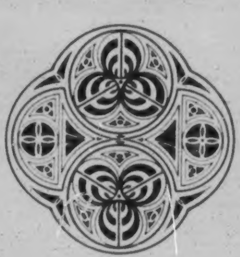


Figure 31

The time saved in class by the projector method enables us to cover more work than most classes do. We review the important ideas of elementary algebra, and learn how to use the sine, cosine, and tangent relations in solving right and isosceles triangles and regular polygons, and four-place logarithms in making computations.

Some theorems, however, are not suitable for this projection method of presentation. When considerable manipulation of equations is necessary, we sometimes set up the required equations on the blackboard and then complete the proof without further use of the figure on the screen. If the proof is long, some students lose their way and occasionally we have to retrace our steps. However, since the room is partially darkened, there is not much to distract their attention and they gradually learn to follow quite long and involved proofs. If a student finds he cannot follow the proofs and is unable to extend his attention span sufficiently, he is allowed to transfer from these classes to a regular class where the work is done in the traditional way.

Figure 32

Each year I have made a few new cards. Now I have accumulated over a thousand. I do not use all of them every year, but I have a fairly wide choice of originals to place before the class. Some of the cards have black lines on white cards; others have white lines on black cards. The former are easier to make, but the latter resemble the figures drawn on the blackboard.

That the plan works is shown by the scores on standardized tests for students who are in the same ability levels in these and in other classes. Two teams of three students each were sent to the Temple University Mathematics Tournament recently and teams from these classes placed first and tenth among the more than seventy teams competing in this tournament.

THE MAKING AND USE OF SLIDES FOR THE TEACHING OF MATHEMATICS

Kate Bell

Some help in the visualization of any problem is necessary if the average student is to understand the subject presented. Teachers of mathematics have always used aids to visualize abstract relationships, but they have not paid sufficient attention to picturing the applications of mathematics in life situations and thus furnishing the link between abstract principles and their function in the environment of the student. A small percentage of brilliant students in mathematics classes really enjoy theory unadorned; but the majority, who are to be the workers of the world, ask with insistence, "What use is it?" They have to be answered, if public education is to reach its goals. Even the slowest students experience some glimmer of understanding when they see pictured the ways in which foresters, aviators, sailors, weather bureau observers, surveyors, and artisans use principles learned in mathematics.

Motion pictures and slides are ideal media for illustrating in a vivid, understandable way the practical applications of mathematics. Motion pictures have their mission in picturing any situation which involves the element of continuous change. For example, locus situations in geometry furnish a rich field. In these cases nothing can demonstrate the idea as vividly as a picture in which the points move. Many students who have failed to get the slightest conception of locus, because they have not enough abstract imagination to see the points move on their controlled journeys, or because teachers have failed to call to their attention the multitudinous and intensely interesting practical applications of the principles set forth in locus problems, have enthusiastically understood after seeing a good expository motion picture.

On the other hand, when continuous motion is not an essential factor, material provided in the schoolroom itself, in the community, and in the immediate region will function better than anything else can in helping pupils to realize the connection between abstract principles and the world around them. Visual aids of the still variety—models, pictures, pupil-drawn diagrams, and slides—have an important place in the effective teaching of mathematics.

Sets of slides may be developed as a cooperative project in mathematics classes. Thirty-five millimeter slides are desirable because the expense is small. Students often have cameras suitable for this work, or the school may have one as part of its equipment. The cheaper cameras produce some excellent results when pictures can be taken at the proper ranges, but for close-up copy work more expensive equipment is necessary.

It is desirable to have a projector of sufficient strength to permit showing pictures in a room only partially darkened. This can be done with a 200-watt projector if the screen is set with its back to the source of the strongest light. A projector designed for larger slides can be used by building a slide holder masked in such a way as to throw the 35 mm. slide in proper position for projection.

The two questions which always arise are, "How is it done?" and "What does it cost?" The first can be answered easily. The second depends on the war, and prices have increased. Slides may be made in black and white or in color. The first step in making a slide in black and white is to take a picture on a 35 mm. negative. From this negative a positive film can be made by exactly the same process as that used in the contact printing of pictures. This positive is put in a mat and bound between two thin pieces of glass. Any number of copies can be made from the one negative. Absolute cleanliness in developing and printing is essential; a dirty or scratched negative is fatal. Because each slide is magnified in a ratio of at least 900 to 1, the most innocent piece of lint on the film or glass looks like rope when the picture is projected. If films are to be commercially developed, they should be taken to some one who guarantees clean film even though the charge is more. Use a soft camel's hair brush to clean the film before printing positives and to clean the positives before mounting in glass. Use soft linen and carbon-tetrachloride to clean film.

Windex or a similar solution may be used to clean the glass on which the film is to be mounted.

Negatives should be kept in glazed paper envelopes and, if positives are not immediately bound in slides, they too should be stored in these envelopes. Positive film is very sensitive and more than usual care is necessary when printing it. Red light must not be used. Exposure to print a positive must be very brief, much shorter than that needed to print pictures, and contact between the positive and negative with emulsion surfaces together must be perfect. Panatomic-X is a good film for black and white work. Many recommend a high contrast film for copy work. If a printing box is used, the light must be weak. Cheap printing frames are entirely adequate. The binding of slides can be done by hand, but a binding machine is a great help. One could cut mats for the slides, but it would be tedious. Because the mats and glass can be re-used when the slides are discarded the expense is slight after the original supply is procured.

Making color slides is a much simpler matter. Kodachrome-A can be used for indoor photo flood pictures and with the proper filter will also do for outdoor pictures, making it possible to shift from one type of work to the other on the same roll of film. One can often take nineteen or twenty exposures on a roll instead of the guaranteed eighteen. These are put in "Ready Mounts" at the processing station. This work is part of the original price of the film. Color slides can be used in this form, but it is better protection to take off the "Ready Mounts" and use glass.

Copy work can be done in both color and black and white. Pictures may be copied from diagrams, photographs, pictures from magazines, and even from newspapers. The last are not as good, of course, because of the imperfections in the newspaper. For such work, auxiliary lenses must be used and careful measurements made. Excellent tables for measurements and settings are furnished with the lenses. A good quality of camera must be used and great care must be exercised in following directions for settings and distance.

For close-up work, a copying stand should be arranged and properly marked for distances and fields, a ground glass being used in the film track to secure focus. When such a stand is once calibrated, copy work will go rapidly and be correct all the time. A homemade copying stand is adequate. The Eastman Kodak Company publishes two helpful booklets, "Copying" and "Slides and Transparencies," which can be secured for 25¢ each. If some other make of camera is used, its manufacturers will furnish you additional information if you request it.

Here are some subjects which have been made into a series of slides:

(a) The work of the United States Weather Bureau, with emphasis on the uses of measurement and of right triangle trigonometry.
(b) The work of the United States Forest Service in locating forest fires and in fighting them.
(c) Geometric design as shown in various forms of art: Indian basketry and beadwork, textile work, ornamental windows, etc.
(d) Geometric forms illustrated by structures of local interest and importance: bridges, buildings, dams, etc.
(e) The history of the development of the airplane, to be used in the new pre-flight aeronautics courses. (Copy work in color.)
(f) The history of mathematics, compiled by copying pictures or having original drawings made by students.
(g) Methods of triangulation and the markers used in surveying. (Copies of U. S. Coast and Geodetic Survey pictures and of student diagrams.)
(h) Application of geometry to problems arising in the construction of the Coulee Dam.

One great advantage in the use of slides is the amount of student-participation possible. Students can help make the slides, arrange them, run the projector, and assist in giving the explanations, and those who are proficient in mechanical drawing and lettering can prepare plates showing diagrams, formulas, and records, of which photographic copies can be made in 35 mm. size. Slides can be prepared from these and interspersed in the proper places in a series of pictures to make clear the applications of mathematics used. Many students know far more about the actual photographic work than does the teacher, and have access to superior equipment. They can help plan series of slides and do the photographic work.

Any teacher can build a reserve of slide material. Students will contribute a great deal, but the teacher should participate; just secure the minimum equipment, read the manuals, question friendly "photography fiends," and experiment. To make early trials more profitable, a record of the range, focus, and exposure for each film should be kept so that the work can be criticized and improved. Nothing surpasses experience guided by reading and advice from those wiser in the art than the experimenter.

INEXPENSIVE HOMEMADE SLIDES FOR DAYLIGHT PROJECTION

Frieda S. Harrell

"SEEING" PERCENTAGE PROBLEMS

IF A COAT is reduced from $48 to $32, what is the rate of discount? Time after time the pupil's answer, 66 2/3%, had to be marked wrong. Why couldn't pupils be made to "see" a percentage problem? In desperation the problem was "drawn" on the board as follows:

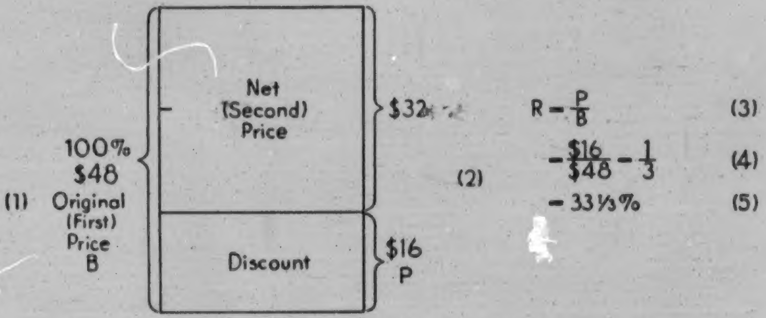


Figure 1

At first this visualization was used only for tricky problems, but later it became more and more useful for the initiation of the percentage concept. Eventually simple slides were tried. An India ink rectangle was projected directly on to the board. Sections of it were cut off in chalk to illustrate per cents. A fourth of it was 25%; three-fifths of it was 60%. A rectangle divided into ten equal sections was found to save time. Later an eight-section diagram was used to teach per cents equal to fourths and eighths. A six-section figure was used to show per cents equal to thirds and sixths. Pupils had then learned the equal parts of 100. They vied

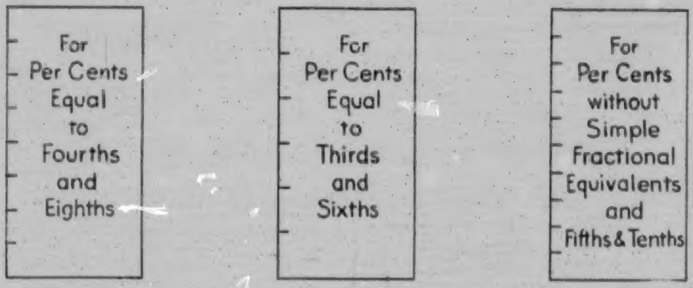

For
Per Cents
Equal
to
Fourths
and
Eighths
For
Per Cents
Equal
to
Thirds
and
Sixths
For
Per Cents
without
Simple
Fractional
Equivalents
and
Fifths & Tenths

Figure 2

with one another in "drawing" the per cents we talked about on the three basic slides shown in Figure 2.

SLIDES ARE EASY TO MAKE

A slide-making program can be started with a small package of cellophane. Cut the cellophane into $3\frac{1}{4}''$ x $4''$ slide mats on which to draw the diagrams with India ink.[1] Graph paper, used as guide lines under the cellophane mat, facilitates drawing and also printing in straight lines. Next, cut some frames with half-inch borders to fit the mats. Laundry cardboards are good for this purpose; oaktag is too thin. Glue the mat between two frames, press the slide under a heavy weight until the glue sets, and try it out. It may be projected in daylight directly on to the board. If the slide proves useful, it can be strengthened with a binding of one-inch glued brown paper available at five-and-ten-cent stores. Figure 3 illustrates simple but complete labeling for rapid selection. The operator's thumbmark at the left-hand side can be easily made with a worn-down pencil eraser and a stamp pad.

If a slide proves to be of permanent value, it can be enclosed between two cover glasses and bound.[2] This binding is narrow and allows extra space for content. A slide holder with an especially large aperture is good. Etched glass can be used for pencil drawings, but it is expensive and unsuccessful for daylight projection. All glass slides require separately applied labels.

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[1] Ready-cut cellophane sells for $1.75 a hundred, with special carbon for typed slides.

[2] Binding for 50 slides costs 15 cents; cover glasses cost about two cents each. J. C. Reiss, 15 Hill St., Newark, N. J., The Keystone View Co., Meadville, Pa., and similar firms carry complete assortments of slide-making materials.

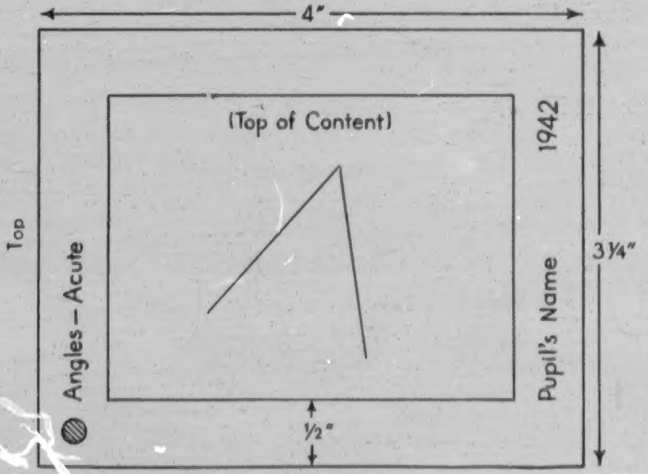


Figure 3

If students are making slides and the cost of cover glass is an important consideration, a dozen glasses can be hinged together in pairs with binding along one side only. Two or three operators, working together, can slip the mats between the pairs of cover glass, project the slides, and take the mats from between the glasses before they are needed again. The mats to be projected for each report can be numbered in the order of projection and submitted to the operators. The operators can be placed one on each side of the projector and one or two behind it. The projector can be focused before the speakers begin. The projection lamp can then be extinguished at the cord switch and the first six mats can be placed between the pairs of cover glass. Two slides can be placed in the projector before the first speaker begins and the third operator can pass the succeeding ones in numerical order to the two side operators. The reporter, when he is ready, can say, "Next slide, please." To signal a pause in projection, he can say, "Thank you," and the projection lamp can be switched off until the next slide is called for. If projection is to be very rapid, the fourth operator can remove projected mats so that the same few pairs of cover glass can be used again. Students quickly learn this operating procedure.

For beginners or for young pupils, slides with cardboard frames may give less trouble. For them, it is wise to cut the cellophane mats larger than 3¼″ x 4″. Pupils can then fasten these larger pieces to book pages, drawings, maps, and so on, with small paper clips. If they fail to center the drawing exactly on the cellophane, the slide can be centered when it is framed. The edges can be cut off after the glue sets and before the slide is bound. For temporary purposes binding is unnecessary.

A newer slide material is lumarith, clear or translucent. The diagram is drawn on this celluloid-like base, which can be used without cover. For permanent protection, the slide can be covered with another piece of clear lumarith and bound with ½″ Scotch tape. Being thin, these slides are easy to operate quickly. They are unbreakable and scratches which develop with use do not show during projection.

More elaborate colored pictures can be made on translucent lumarith or on etched glass to be shown in a dark room, but the advantages of daylight projection cannot be overestimated. Colored India inks or slide inks can be used for small surfaces on cellophane or for larger surfaces on clear lumarith. These will show in daylight, even on the board. All colors used must be transparent. Slides made from cut-paper silhouettes are very effective, especially for auditorium programs. Enclose them between lumarith covers washed with colored slide inks. Several thicknesses of colored cellophane can also be used.

WHEN A SLIDE IS VALUABLE

Slides are superior to mimeographed material because the students' attention can be concentrated on one particular problem or paragraph at a time. When one small area in the room is brightly lighted, attention can be focused there. Slides prevent the lapse in class attention which usually occurs while the teacher turns to draw a difficult or accurate diagram on the board. Complicated demonstration diagrams can be drawn to scale, and then traced on slides. Constructions can be completed in chalk after the diagram is projected on to the board. Board space can be saved where many classes or many subjects must be taught in the same room. Almost any teaching material worth preserving for successive sections of students or for successive years of teaching justifies the work of putting it on slides.

CONTENT MATERIAL FOR STUDENTS' SLIDES

Presentation of informational material by students can be motivated if slides are used. The reporter can print or type his outline on a slide. Unusual or new words can be listed on another slide. Both slides can be left in the slide holder and shown several times during the report. The visual stimulus then supplements the auditory one without the need of the reporter's writing on the board. Simple illustrations, graphs, maps, or cartoons can be drawn or traced on cellophane. Tracing paper can be used first when material is being copied from valuable books. For this work teacher and students can collect and preserve slide-size material from magazines and newspapers. Students will make these simple black and white diagrams by the dozen; the difficult task is to encourage them to make fewer but better ones. Small irregularities can be erased with dampened cotton twisted over a stick pointed in the pencil sharpener. However, even a rough illustration frequently drives home a point. A student who has little artistic ability need not be held to a standard too high for him. Auditorium slides, since magnification is high, must be carefully made. These slides rarely fail to supply the inarticulate student with plenty to say. Historical topics take on new interest if the reporter uses handmade slides. He can make simple pictures of Greek, Hindu, or Chinese numerals. He can show clocks for a report on the history of time. He can explain Roman addition, adding chalked answers to projected examples. A blank form, projected, can quickly be transformed into a magic square or circle if the investigator adds chalked numbers. Any slide which can be completed quickly in chalk is popular with the audience. Mathematical riddles mean more if the audience can study them from a slide. Mathematical tricks and puzzles can often be explained more easily from slides than by other means.

SLIDES FOR DRILL WORK

Slides are particularly useful for drill work, for summaries, or for short, rapid reviews of facts. For example, lines, angles, triangles, flat figures, and parts of circles can be quickly illustrated, each on a separate slide and each in several positions. The students can number their papers, and as each slide is projected and its number called, they can name the figure beside its number on their papers. The slides can then be reprojected in the original order as the students correct and discuss their errors in recognition and spelling.

Slides can supplant such traditional mimeographed testing material as true-false or completion exercises.

PROJECTED GRAPH "PAPER"

Projected graph "paper" has several advantages over the familiar squares painted on one section of the blackboard. Board space is released for other work when graphs are not in use. Several types and sizes of square can be employed in the same lesson. Widely differing values of a variable can be shown in the same amount of space. Bright yellow chalk on projected black squares is more readily visible than any other color of chalk on painted squares. The slides can be made over graph paper by hand. With a fine pen, the horizontal lines can be drawn on one side of the cellophane and the vertical lines on the other. This method allows small erasures. Curling of the cellophane does not affect the projection.

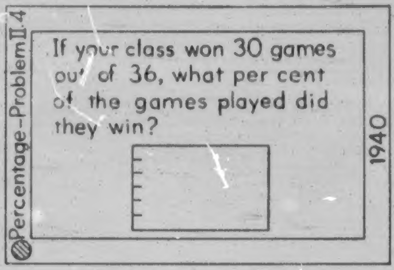
SLIDES FOR AUDITORIUM PROGRAMS

Slides are useful for auditorium programs. The announcer can give place to a series of typed slides. Speakers' names are more intelligible if read from slides, a separate temporary slide for each. Unusual words, each on a temporary slide, can be projected for a few seconds as they are used by speakers. Maps can be projected as place names are mentioned. The back wall of the stage can be used if the screen is in the way of performers. A small screen can be set on a table and slides can be projected from a convenient spot on the auditorium floor.

PERMANENT SLIDES FOR A UNIT OF TEACHING

When a unit must be taught year after year, time can be saved by developing it completely on permanent slides. It can be kept up-to-date by the addition of a few slides each year. The topic on percentage, mentioned at the beginning of this article, was so treated.

After practice on the three basic slides, it seemed advisable to use the diagram for almost all problems. It was particularly useful to show distinctions like 25% *of* a number, 25% *more than* a number, and 25% *less than* a number. Later, demonstration problems were typed or printed in India ink on the same slide as the appropriately sectioned diagram, as shown in Figure 4.



If your class won 30 games out of 36, what per cent of the games played did they win?

Figure 4

Completion of these problems on the board resolved them into five "thinking steps" (numbered in Figure 1). Inter- and intraclass contests were organized, each student calling on another from the opposing team, and each completing one of the thinking steps for a point. Much more concentrated attention can be secured if only one problem is visible at a time.

Later the class ruled unsectioned rectangles on their own papers and diagrammed problems from their exercises, roughly estimating where the dividing line would fall. For instance, they would locate the 50% and 25% positions with tiny marks. Then a 39% line would be placed closer to the 50% mark, while a 36% line would be placed closer to the 25% mark.

When second-case percentage was taught, the whole diagram was marked not only 100% but also with the quantity that represented 100%, the base, as in Figure 5. Students were now trained to locate a line representing *P*, \$12 out of \$15, 10 games out of 15 games, 16 points out of 20 points. Many simple second-case problems could be answered immediately from the diagram. Sample diagrams and solutions follow:

> John started his flock of pigeons with 16 birds. Now he has 325% more. How many pigeons has he now? (See Fig. 5.)

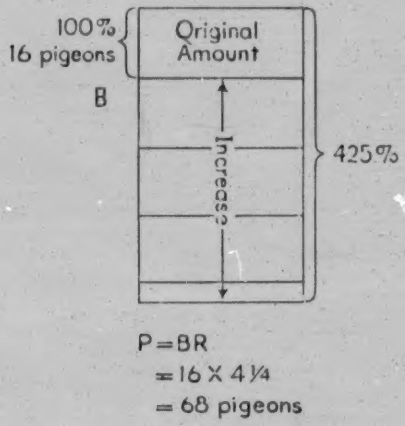


Figure 5

> If you solve 19 problems out of 20, what per cent of your problems do you have right? (See Fig. 6.)

Many other slides can be used to avoid monotony. A rectangle containing twelve squares can represent a dozen oranges. A sixteen-square diagram can represent a pound. A circle divided into sectors can represent a chocolate cake. A hundred-section square is useful. A parallelogram can be divided into 50% sections in four different ways.

These rectangular graphs can be used to visualize percentage without slides. However, slides save class time to such an extent that the teacher who makes them is amply compensated for the work involved.

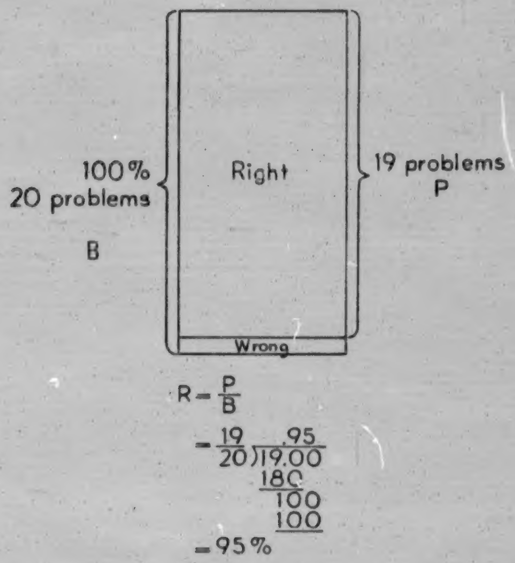

100%
20 problems
B
Right
Wrong
19 problems
P
R = P/B
= 19/20
.95
20)19.00
180
100
100
= 95%

Figure 6

Price List for Slide Making Materials

Cover glass	
Clear	$ .25 per doz.
Etched	$1.25 per doz.
Binding	
Glued black paper for 50 slides	$ .15 per package
½" Scotch tape for 86 slides	$ .65 per roll
Cellophane slide mats	
Plain ready cut cellophane with carbon	$1.75 per 100
"Radio" mats (cellophane mat folded inside carbon and masked)	$1.50 for 50
Lumarith slide mats, etched and masked	$6.00 per 100
Sheets of lumarith	
20" x 16¼"	
Clear	$ .65 each
Etched	$ .85 each

20″ x 50″

Clear	$2.00 each
Etched	$2.50 each

25″ x 40″

Clear only; .010″ thickness.
Orders of 200 sheets or more are accepted by the Celluloid Corporation, 290 Ferry St., Newark, N. J., at $.442 per sheet. (Add 25% to this price for smaller quantities. Orders amounting to less than $10 not accepted.)

Separate slide masks	$1.00 per 100
Slide crayons	$ .90 for 7 colors
Slide inks	$2.25 for 7 colors
Slide ink solvent	$ .30 per can

Ordinary colored writing inks or India inks may also be used if they are transparent.

A LANTERN-SLIDE PROJECTOR AS A CONSTRUCTION PROJECT

Gordon MacLeod Taylor

THE AVERAGE classroom teacher has no time to go to distant parts of the school building to obtain equipment. For that reason, unless a slide projector is provided for each classroom, ready for use whenever it is needed as is the pencil sharpener, the chances are that very little use will be made of slides as a teaching aid.

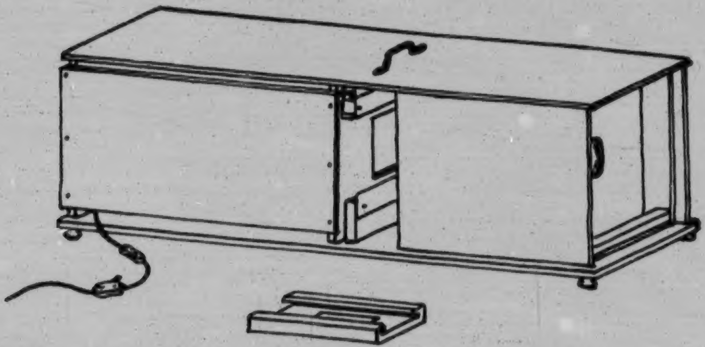


Figure 1. The Projector

Obviously the cost of the lanterns precludes the possibility of obtaining a large number of machines. If, on the other hand, a reasonably satisfactory device could be produced at a nominal cost, there would be little need for any teacher to be deprived of such a tool.

To meet these requirements, this article describes a projector (see Figure 1) which can readily be built by a junior or senior high school student. The materials required have been confined to those readily obtainable in any community, while the design has been simplified in order to avoid complicated measurements and the use of any except the most common tools. As an educational activity alone, the construction of such a device would easily justify the expenditure of the less than five dollars which will provide all needed materials.

This projection lantern will give reasonably satisfactory results in a semi-dark room. In a totally dark place the pictures will be surprisingly clear and brilliant. The machine is planned to be used with either opaque or transluscent screens and, by the substitution of suitable attachments, various types of films as well as slides can be shown. The design is such that no extra carrying case is needed when the machine is not in use, as the lens mount can be pushed in far enough to close the operating openings in the sides of the body and leave the focus chamber available for the storage of slides and attachments.

	Materials	*Approx. Cost*
1	piece ⅜″ plywood, 36″ x 24″	65¢
1	piece ¼″ plywood, 48″ x 24″	65¢
1	piece ¾″ white pine, 10″ x 8″	10¢
50	½″ brads	10¢ a gross
50	¾″ brads	10¢ a gross
18	½″ #4 round head blued screws	35¢ a gross
2	¾″ #8 round head blued screws	1¢
1	tin can, at least 10″ in circumference	
1	tin can, round, gallon size	
1	flat base keyless porcelain socket	10¢
4	yards lamp cord (preferably heavy duty rubber covered type. Get length you need)	40¢
1	cord plug cap (rubber body type preferred)	10¢
1	indoor floodlight bulb, 150 watt, type R-40	95¢
1	reading glass (2½″ dia. from dime store) (A good quality 4″ dia. lens at $1.75 will give much better results)	25¢

The following may be added if desired:

1	two-piece appliance cord connector	20¢
1	feed-thru appliance cord switch	25¢
1	sash lift or drawer-pull of the handle type	10¢
4	knob type, felt-tipped feet	20¢

Tools

Yardstick or ruler
Try square or steel square
Pencil
Crosscut saw
Ripsaw

Coping saw frame and blade
Brace and ⅜" bit
Expansion bit, ⅞" to 3" (Note: If the lens to be used is over 3¼" in diameter, this bit will not be needed, as the hole in the lens holder board will have to be cut with the coping saw.)
Hand drill
⅛" Twist drill (Note: An 8 d. nail with the head cut off will serve as such a drill, and if no hand drill is available, may be used in the brace.)
Plane
Hammer
Screw driver
Penknife
⅛" Sheet metal punch or 8 d. nail cut off flat
Can opener
Tin snips
#½ Medium sandpaper
1" Paint brush
Black enamel
Liquid glue

PREPARATION OF THE PARTS

Step 1. From ⅜" plywood, cut the following pieces:

1 piece	33¼" x 10"	Base of Machine
2 "	9¼" x 8½"	Lamphouse Ends
1 "	8⅝" x 7⅞"	Lens Holder Board

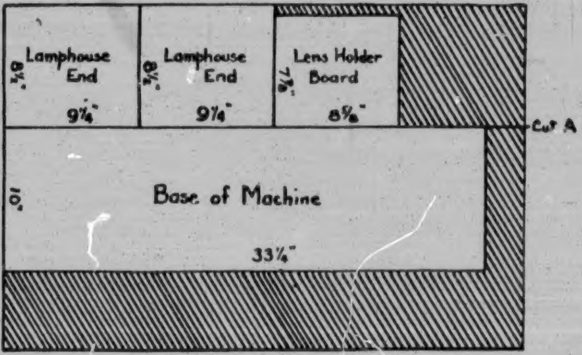


Figure 2. Parts to Be Cut from ⅜" Plywood

Figure 2 suggests a layout which will allow all pieces to be cut from the smallest standard-size panel. Use the crosscut saw and start by making "Cut A." Instead of marking out the whole board, measure and mark the line for each cut after you have made the previous cut. When sawing, keep the saw blade on the outside of the line so that the saw cut will not take away from the size of the piece you are preparing.

Step 2. Mark off one of the Lamphouse Ends as shown in Figure 3. Use a ⅜″ bit in the brace to drill a hole as shown. Insert the coping saw blade through this hole and attach it to its frame. Saw out the 3¼″ x 4″ hole as marked out. This piece with the hole in its center will be the Front Lamphouse End.

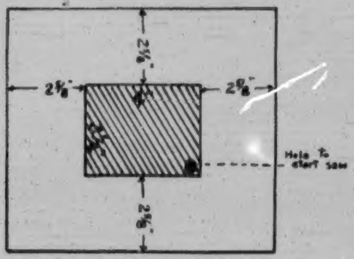


Figure 3. Detail of Front Lamphouse End

Step 3. Draw diagonal lines from corner to corner of the Lens Holder Board. Adjust the expansion bit to drill a hole ¼″ smaller in diameter than the diameter of the lens you are going to use in your projector. Lay the plywood upon a piece of scrap lumber an inch or more thick. Placing the point of the bit on the spot where the diagonal lines cross, drill completely through the plywood. As the center of the bit will go through before the extension cutter, it will be necessary to hold the plywood to the scrap lumber at all times in order to have the large hole cut clean.

Step 4. From ¼″ plywood, cut the following pieces (see Fig. 4):

1	piece	33″ x 9¾″	Top of Machine
2	"	17″ x 7¼″	Lamphouse Sides
2	"	11¾″ x 8½″	Focus Chamber Sides
2	"	8⅜″ x 4¾″	Lens Mount Sides
2	"	8⅝″ x 4″	Lens Mount Top and Botton
1	"	8″ x 4½″	Slide Holder Back
2	"	1¾″ x ¾″	Slide Holder Latches

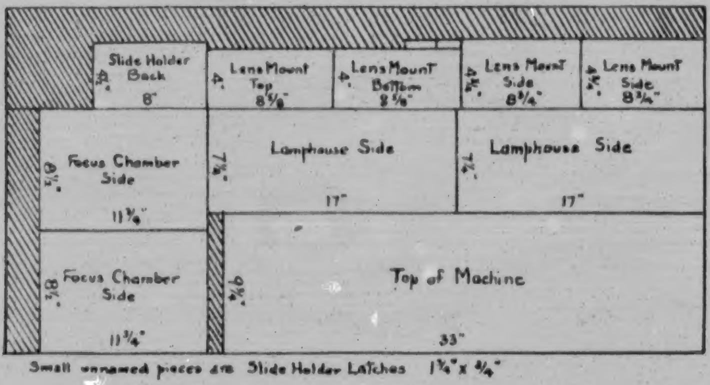

Slide Holder Back
8"
Lens Mount Top
8⅝"
Lens Mount Bottom
8⅝"
Lens Mount Side
8¾"
Lens Mount Side
8¾"
Focus Chamber Side
8½"
11¾"
Lamphouse Side
7⅝"
17"
Lamphouse Side
17"
Focus Chamber Side
8½"
11¾"
Top of Machine
9¾"
33"
Small unnamed pieces are Slide Holder Latches 1¾" x ¾"

Figure 4. Parts to Be Cut from ¼" Plywood

Step 5. Using the brace, ⅜" bit, and coping saw as in Step 2, cut the 3" x 3¾" hole in the Slide Holder Back, as shown in Figure 5.

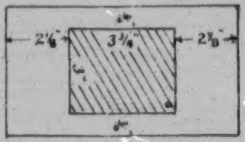


Figure 5. Detail of Slide Holder Back

Step 6. Using the brace and ⅜" bit, drill a hole in each Slide Holder Latch, as shown in Figure 6.

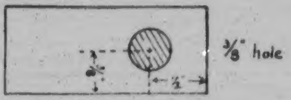


Figure 6. Detail of Slide Holder Latch

Step 7. Using a ⅛" twist drill in the hand drill, make the six holes in one of the Lamphouse Sides, as shown in Figure 7 (on the next page).

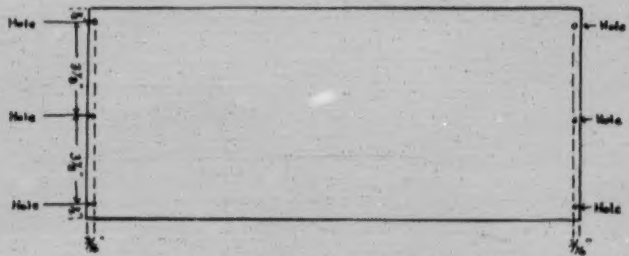


Figure 7. Detail of Right Lamphouse Side

Step 8. From ¾" white pine, cut the following pieces:

1 piece	8" x 2" scant x ¾"	Upper Slide Holder Guide
1 "	8½" x 2" scant x ¾"	Lower Slide Holder Guide
1 "	8⅝" x ¾" x ¾"	Lens Mount Handle
2 "	8" x 5/16" x ½"	Slide Guides (see note)
2 "	8" x ¼" x 5/16"	Slide Guide Strips (see note)

Note: If a power saw or rabbeting plane is available, the Slide Guides and their Strips may be made as single pieces 8" x 9/16" x ½" with a ¼" x 3/16" rabbet. Figure 10 indicates the position of the rabbet.

In cutting these pieces, use the cross-cut saw when cutting across the grain and the ripsaw when cutting with the grain. To have the pieces exactly the right size, it would be well, when using the ripsaw, to cut the pieces somewhat large and then plane them to size.

Step 9. Remove both ends of the small tin can. With the snips cut the curved part of the can along the seam and flatten out the resulting sheet of metal. Mark and cut out the two Slide Holder Guide Plates, using the patterns shown in Figure 8.

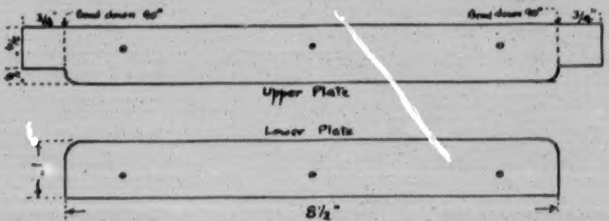


Figure 8. Detail of Slide Holder Guide Plates

Place the plates on a piece of scrap lumber and make the indicated holes by pounding the sheet metal punch through them.

Turn the plates upside down and lightly tap the metal around the holes to flatten the burred edges.

Now lay the Upper Slide Holder Guide Plate on the 8½" long Lower Slide Holder Guide made in Step 8. Put it in such a position that the ¾" wide portions of the plate project beyond the ends of the wooden guide. Bend these metal tabs down at right angles to the main part of the plate.

Step 10. From the remaining sheet metal, cut four pieces ¾" x ½". Punch a hole in each, ¼" from the end. These pieces are the Lens Clips which will be used to clamp the lens to the Lens Holder Board.

Step 11. Remove one end of the gallon can and smooth the edges so that there are no sharp parts to cut the hands. This can is to form the Heat Baffle which keeps the hot rays of the bulb from reaching the casing of the projector.

Mark a 3¼" x 4" rectangle in the center of the bottom of the can. Pierce the blade of a strong penknife through the metal inside of the rectangle. By pulling the blade nearly out of the hole and then pushing it down again while the cutting edge is being pressed against the metal, the rectangle may gradually be cut out. The edges of the metal will be bent inward, but they can easily be straightened out by placing the can upside down over a piece of wood long enough to reach up into the can and press against the inside of the bottom while the outside is lightly tapped with a hammer. While the can is in this position, punch the two screw holes shown in Figure 9, tapping the edges smooth afterward.

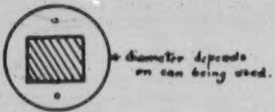


Figure 9. Detail of Heat Baffle End

Step 12. With a piece of sandpaper folded over a block of wood, smooth the edges of all the pieces of wood you have prepared. Be careful that you do not round the edges; merely smooth them, leaving their corners square.

Paint all sides and edges of the wooden parts. Except for the Heat Baffle, the metal parts should also be painted. Allow them to dry completely. If necessary for a good finish, a second coat of paint may be applied.

ASSEMBLING THE PARTS

Step 13. The Slide Holder. Spread a thin coat of glue along one 5⁄16″ wide side of the Slide Guide Strip (from Step 8). Place the glued side down on the ½″ wide side of one of the Slide Guides in such a position that one edge of the narrow stick is along the edge of the wide one. Be sure the ends are even; then nail down with four ½″ brads. In a similar manner, assemble the other Slide Guide Strip and Slide Guide. (See Fig. 10.)

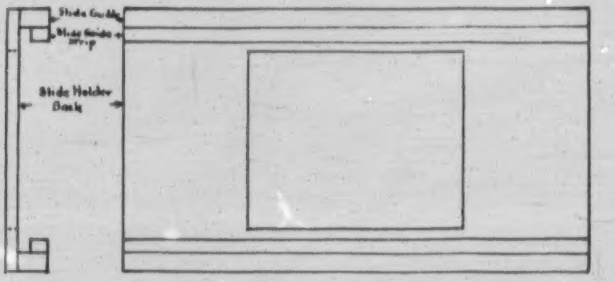

Slide Guide
Slide Guide Strip
Slide Holder Back

Figure 10. The Slide Holder

Now apply glue to the 5⁄16″ wide side of the Slide Guides and place them along the edges of the Slide Holder Back (from Step 5). Secure each by driving four ½″-brads through the back into the guide. This completes the Slide Holder.

Step 14. The Lamphouse Front Assembly. Apply glue to the 2″

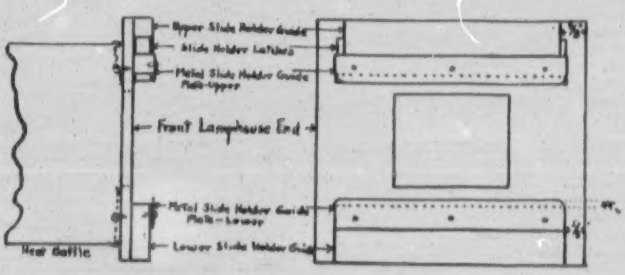

Upper Slide Holder Guide
Slide Holder Latches
Metal Slide Holder Guide Plate-Upper
Front Lamphouse End
Metal Slide Holder Guide Plate-Lower
Lower Slide Holder Guide
Rear Baffle

Figure 11. The Lamphouse Front Assembly

sides of the Slide Holder Guides (from Step 8). Lay each with its edge along one of the 9¼″ edges of the Lamphouse Front (from Step 2) and equally distant from the ends. (See Figure 11.) In this position, the Slide Holder (from Step 13) should slip freely but not loosely between them. Leave the holder in this position while completing Step 14. Secure the Slide Holder Guides in position by driving four ¾″ brads through the Lamphouse End into each of them.

Place the Lower Slide Holder Guide Plate (from Step 9) upon the Lower (8½″) Slide Holder Guide with the ends of the plate and the guide even. The upper edge of the plate must extend ¼″ beyond the guide to form a lip to hold the Slide Holder in place. (See Figure 11.) Fasten the plate in position with three ½″ screws.

Fit one of the Slide Holder Latches (from Step 6) to the end of the Upper (8″) Slide Holder Guide. The hole should be in the end of the latch nearest the Slide Holder. Adjust the latch so it extends ¼″ below the edge on the Slide Holder Guide. In this position, when the assembly is complete the latch will prevent the removal of the Slide Holder. With the latch thus placed, drive a ¾″ brad through the hole in the latch, keeping the brad as far away from the Slide Holder as the hole will permit. Pound in the brad until its head is level with the latch. Now the latch can be slipped up until it clears the edge of the Slide Holder, which may then be pulled endwise from its position between the guides, or the latch may be slipped down until it overlaps the end of the Slide Holder by ¼″. The Upper Slide Holder Guide Plate will hold the latch to the end of the guide in the completed assembly. Fit the other latch to the opposite end of the Upper Slide Holder Guide in the manner just described.

With both latches in place over their brads, lay the Upper Slide Holder Guide Plate upon the Upper Slide Holder Guide as was done with the lower guide and plate. The lower edges of the bent tabs of the plate should be even with the bottom of the guide, thus allowing a lip of the plate to project downward ¼″ to hold the Slide Holder in place, as was the case with the plate on the lower guide. Test to be sure the position of the plate is such as to allow the Slide Holder to be withdrawn endwise when the latch is raised, but to hold it in its proper place when the latch is down.

If necessary, the plates may be bent slightly to make the holder move more freely. When sure that the parts are in place, secure the plate by means of ½″ screws.

Now turn the assembly guide-side down on the workbench. Place the Heat Baffle (from Step 11) bottom to Lamphouse End, center the hole over the opening in the plywood, and fasten it in place with two ½″ screws.

Step 15. The Lamphouse Rear Assembly. Draw diagonal lines from corner to corner of the Rear Lamphouse End (from Step 1). Using ¾″ screws, fasten the porcelain socket to the plywood. The diagonals will make it possible to get the center of the socket in the exact center of the plywood.

Cut a 9″ length from the lamp cord.* Separate the two wires it contains for a distance of 2″ from one end. Scrape the insulation from ½″ of the end of each of these wires. Screw each down tightly under one of the terminals of the socket. Tie the cord in a knot as close to the socket as is convenient, and with a piece of string through the knot, tie the lamp cord to one of the screws that holds the socket to the board in such a manner that any pull on the lamp cord will not pull the wires from under the socket terminal screws.

Separate the wires in the other end of the 9″ cord for a distance of ½″ and scrape the insulation from ¼″ of their ends. Take apart the half of the Appliance Cord Connector which has the projecting blades. You will find a terminal screw on the inner end of each blade. Attach one of the bared ends of the wires to each terminal, and then put the connector together again. Be careful that none of the little strands of one of the wires is able to get to bare metal connected to the other wire.

Make up the long lamp cord by connecting the other half of the Appliance Cord Connector to one of its ends and the Plug Cap to the other end. Make these connections as you did the half of the connector on the short cord, being careful about the little wire strands as before. A few inches from the connector on the long cord, separate the two wires that make up the cord. They should be separated for only about an inch along the length of the cord. In the middle of the separated place cut one of the wires in half.

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* ***Note:*** **If the connector and the switch are to be omitted, the cord is connected directly to Socket and Plug Cap, no 9″ piece being cut from it.**

Scrape the insulation from ¼″ of these cut ends. Open the Feedthrough Cord Switch. Connect the two bared wire ends to the terminals in the switch and lay the uncut wire in the channel provided to carry it through the switch. Reassemble the switch.

Step 16. The Lamphouse. The lamp cord leaves the socket on the lower side of the Lamphouse Rear Assembly. Apply glue along this 9¼″ edge of the plywood. By means of four ¾″ brads driven through the base of the machine (from Step 1) attach the Lamphouse Rear Assembly to the base, placing it so that the base extends ⅛″ behind it and ⅜″ on each side of it. Obviously the socket is on the side which will be inside the lamphouse.

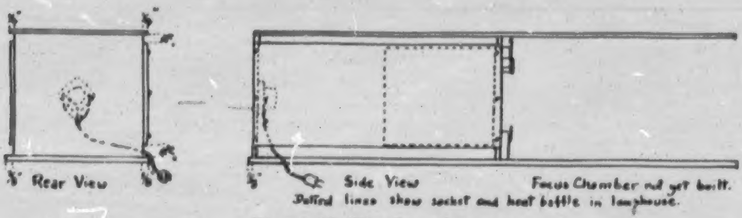

Rear View
Side View
Focus Chamber not yet built.
Dotted lines show socket and heat baffle in lamphouse.

Figure 12. The Lamphouse

Facing the rear of the lamphouse, apply glue to within about an inch of the top and bottom of the left edge of the Rear Lamphouse End. By driving ¾″ brads through the Left Lamphouse Side (the one without holes) fasten it to the glued edge of the Lamphouse End. It should be placed so that there will be a ¾″ space between its lower edge and the Base of the Machine.

Apply glue to the bottom and left edges of the Lamphouse Front Assembly in a similar manner. Remember the Slide Holder Latches are at the top of the assembly and the Heat Baffle extends back into the lamphouse. Drive brads through the front end of the left side panel to secure the front assembly in place. Be sure that the front edge of the side panel does not project beyond the plywood surface of the front assembly or it will interfere with the removal of the Slide Holder.

By means of six ½″ screws, fasten the Right Lamphouse Side in place to correspond to the left side which is glued and nailed on. The right side is attached with screws to permit its removal for the installation of the bulb.

With both sides in place, the front is now held in true position and may be fastened to the base by means of four ¾″ brads driven up into it. Like the rear of the lamphouse, it is ⅜″ from the sides of the base.

Apply glue to the tops of the lamphouse ends and secure the Top of Machine (from Step 4) in place by means of eight ¾″ brads. The rear of the top is even with the rear of the lamphouse end, but its sides extend ¼″ beyond the sides of the ends. The ¾″ spaces below and the ½″ spaces above the side panels are to provide ventilation for the bulb.

Step 17. The Lens Mount. Apply glue to both ends of the Lens Mount Bottom (from Step 4). By driving three ¾″ brads through each of their ends attach the Lens Mount Sides to the glued edges of the bottom to start forming a boxlike assembly. Since the sides are wider than the bottom, one of their edges will extend beyond the edge of the bottom, but the other edges of all three should be even. This side of the assembly will be the front of the finished Lens Mount. (See Fig. 13.)

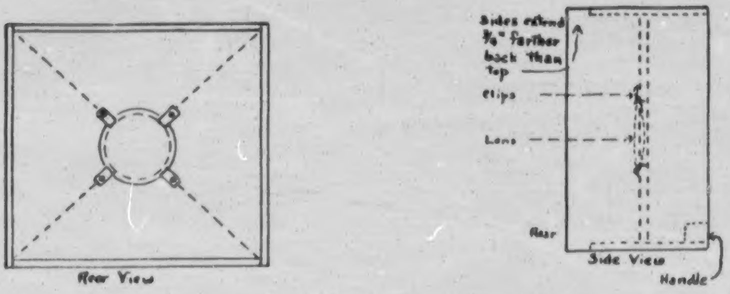


Figure 13. The Lens Mount

Apply glue to the bottom (8⅝″) edge and the two side edges of the Lens Mount Board (from Step 3). Place it in the box already prepared, in such a manner that its surface without diagonal line markings is two inches from the even front edge of the box. Secure it in place with six ¾″ brads.

Apply glue to the upper edge of the Lens Mount Board and to the two ends of the Lens Mount Top. Complete the boxlike mount by putting the top in place to correspond to the bottom.

Apply glue to both ends and one of the sides of the Lens Mount Handle (from Step 8). Place it in the mount along the front bottom edge and secure it in position by means of ¾″ brads through the sides and bottom of the mount.

Place the mount front down on the workbench. Remove the reading glass lens from its frame. Lay the lens on the Lens Mount Board. Center it over the hole and, using the four Lens Clips (from Step 10), clamp it in place. The clips should be bent so that when the ½″ screws through them are tight, there will be pressure enough to keep the lens from rattling, but not enough to crack it.

Step 18. The Focus Chamber. Apply glue to the long edges of the Focus Chamber Sides (from Step 4) and place them between the top and bottom of the machine. Their outer sides should be even with the edges of the top of the machine, but ⅛″ in from the edge of the sides of the base. Their front edges set ¼″ in from the front edge of the top of the machine and ⅜″ in from the front of the base. Secure them in place by means of twelve brads through the top and base. Their position should be such that the lens mount may freely, even loosely, move within the chamber.

Step 19. Miscellaneous. Remove the lamphouse side and put in the bulb. Attach the carrying handle to the top of the machine in such a position that its screws go down into the upper slide holder guide. Screw the feet to the four corners of the base, and the machine is ready for use.

OPERATING THE PROJECTOR

The machine is operated in the usual manner. Any smooth, light-colored surface may be used as a projection screen. The darker the room and the smaller the image on the screen, the brighter the projection will be. The size of the image at any given distance will depend upon the lens. A 2½″ diameter dime-store reading glass lens with a focal length of about 8¼″ gives a picture three feet wide when the machine is about nine feet from the screen. If it is desired to use the machine for projection through a translucent screen with the machine 52 inches behind the screen, a second similar lens may be clamped to the front of the lens mount board and the resulting picture will still be about three feet wide, although, because of the type of lens, its shape will be somewhat "pincushioned."

Because the projector is built without expensive condenser lenses, the screen illumination is not even over the entire surface. However, the dimming-out at the corners is not too objectionable with photographic slides, while it is practically absent when ground glass or frosted lumarith is used for homemade slides.

THE GEOMETRY TEACHER EXPERIMENTS WITH MOTION PICTURES

Rachel P. Keniston and Jean Tully

THREE YEARS AGO, Professor Eby of the College of the Pacific in Stockton, California, developed a course in visual aids for the purpose of inspiring classroom teachers to experiment with them and to use them in practical ways. The writers enrolled in this course which was composed mainly of teachers. The group were enthusiastic to find the variety of visual aids that could be inexpensively made for all types of classes from the primary grades to college level. Much of the time was spent in considering individual problems; consequently, members of the class were free to concentrate on their own projects.

After considering various projects applicable to mathematics, such as slides for projection of special problems or tests and the still filmstrip, we decided to make a geometric moving picture as our project, although neither of us knew anything about taking pictures or developing them. In this article we shall relate our experiences in making two films.

We chose as our first subject "Locus in Plane Geometry" because the subject matter seemed particularly fitted for motion pictures. Our second subject was "Concurrency in Triangles."

The locus film begins with six locus theorems, usually considered fundamental. Each of these we showed first as geometrically constructed, and followed by applications taken for the most part from some life situation and presented by means of an animated cartoon. Then, to give a glimpse of more complicated aspects of locus, we pictured some compound loci, the cycloid, and the conic sections. Thus the film is one that can be used with success in a class after their initial work, serving to humanize and clarify the subject, to present a unique review, and to give the students a desire to know more about the topic.

The film on concurrency developed the four cases of the concurrency of lines in all types of triangles. Questions dealing with the position of the point of concurrency were presented for the student to consider and were intended to lead him to definite conclusions concerning concurrency and the influence upon it of the type of triangle and lines involved.

These pictures were taken entirely indoors from drawings made on paper 20″ x 27″. The drawings were attached to a drawing board of about the same size, which in turn rested on a table that was set in a groove in the floor to prevent change of position. Stand lights may be used, but we used a rectangular frame of photoflood lights around the edge of the table and controlled it by a foot trip. If stand lights with four number 2 photofloods are used, they can be placed farther away from the board.

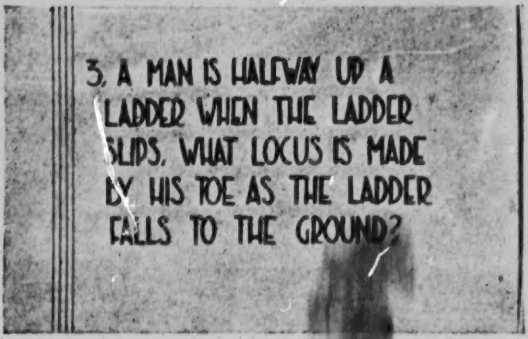
The camera was held in place by a frame attached to the wall in such a way that the camera was directly above the drawing. Professor Cox at Oregon State College had originally suggested building such a stand out of an electric drill press, thus making the camera highly adjustable, yet rigid with respect to the drawing board.

When we were shooting titles, the camera was placed four feet from the board. Figure 1 shows a title used in this picture. The lines of the lettering should be about 3/16″ in width. The title shown measures about 7″ x 13″, and, when projected, fills the same space as the diagrams.

For shooting diagrams, the camera was placed six feet from the drawings. Thus larger drawings could be used and lines on them could be thick enough to project well. Main lines on drawings were about ¼″ wide.

Various devices were used to make the animation and explain the continuity. Titles and running comments were printed as illustrated in Figure 1. We used black drawing ink on white drawing paper, having discovered after experimenting that white paper gave better results than yellow. For both cartoon and construction work, the basic lines of each setting were drawn first, and the parts that moved were added against this background.

Figure 2 shows the cartoon used to accompany the locus described in the title in Figure 1. Here the house and scenery were first drawn on paper. We used a brush, thus gaining a better effect



3. A MAN IS HALFWAY UP A LADDER WHEN THE LADDER SLIPS. WHAT LOCUS IS MADE BY HIS TOE AS THE LADDER FALLS TO THE GROUND?

Figure 1

with suggested lines than we could have done by attempting an accurate drawing. The man and the ladder were made on separate pieces of paper. To obtain the motion of the ladder's falling to the ground, the ladder was moved about an eighth of an inch at a time, and about two frames were snapped for each position. The dots were pasted along the locus as the path was made. Then a compass was used to trace the arc in order to point out the locus theorem involved.

There were, we found, many pitfalls in working with anima-

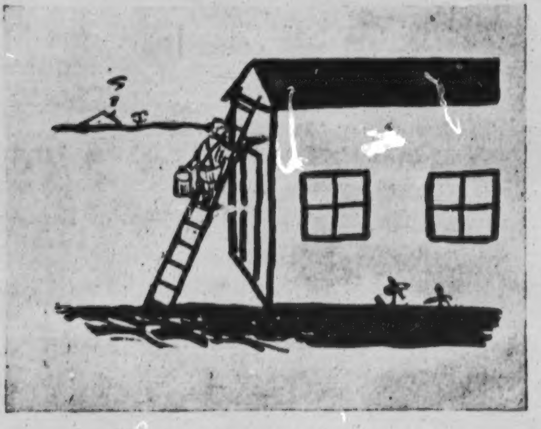


Figure 2

tion. For example, we wished to illustrate the simple circle locus by showing the boundary of the area over which a cow may graze if tied by a rope to a stake. Having painted the background, we discovered that it was impossible to move a toy cow around the locus, as planned, for the camera from above would show the *back* of the cow as in an airplane view, and this would not coordinate with the angle at which the scenery was drawn. This problem was solved by making several cardboard models of a cow and laying them flat on the drawing in many of the progressive positions that she would take on her trip around the stake. Then, as the locus was traced by means of successively added dots, one of these cows was always present. The final effect was quite satisfactory, except for the slight leaps of the cow due to the too great distance between the positions of the models.

Another illustration of this circle locus was unsuccessful. We tried to show the locus made by the tip of the hour hand of a clock. We used an alarm clock placed on a piece of drawing paper. We snapped two frames, moved the hand an eighth of an inch, snapped two more frames, and repeated the process until the hand had moved completely around the dial. It took a long time to make the picture. Unfortunately, it was humanly impossible to set the clock in exactly the same spot each time the hands were moved; as a result the picture showed a very lively dancing clock, with the hour hand breaking all speed records on its journey around the dial. It was an excellent illustration of the way *not* to show animation.

Our applications of locus were not always outside the field of geometry. For example, to illustrate the theorem *The locus of points equidistant from two points is the perpendicular bisector of the segment joining them*, we used two purely geometric settings, namely the locus of the vertices of isosceles triangles on the same base and the perpendicular bisector of the chord of a circle. In the first, black and white cardboard triangles of various altitudes were moved gradually into the picture coming to rest on the common base which had been previously drawn on paper. In the second, rulers were used to locate the points equidistant from the ends of the chord and dots were made with a die to show the locations. We followed the natural way to consider a locus, by locating several points to satisfy the condition and then drawing the locus, which was shown by the motion of the point.

We included in our illustrations touches we hoped the student would find humorous. For example, to illustrate the locus of points midway between two parallel lines, we showed a fly wandering around a room and finally taking a direct path across the wall. The fly, the flit gun which ended his life when it stealthily appeared in the corner of the picture, and the footprints he left—incidentally recording the locus—were cut from paper and applied to the drawing. Here again, a sufficient number of frames were taken of each position to make the final motion slow enough to follow.

In making purely geometric applications and in showing constructions, particularly in the film on concurrency, more mechanical devices were needed. Figure 3 shows some of the objects used.

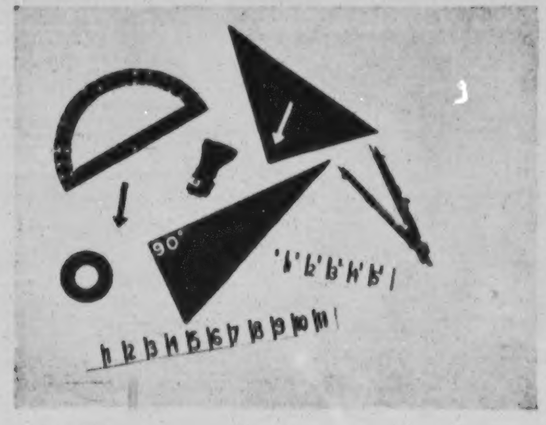


Figure 3

The compass was a metal one. The stamp was our most frequently used implement, because it made a dot one-fourth inch in diameter and all construction lines were shown by a succession of such points added manually. All other articles, such as the protractor, rulers, and triangles, were made of cardboard and marked with black ink. The protractor and the rulers were used to show measurements and to give variety to the pictures. The arrows were very convenient to point out things to be particularly noticed.

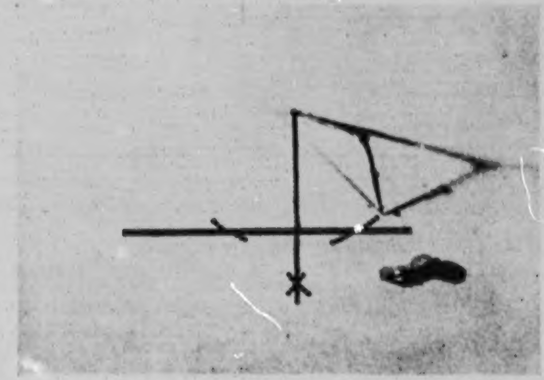


Figure 4

Figure 4 shows a simple construction as it is just being completed. The way a compass can be used is especially interesting. The point of the compass was removed, and the compass screwed over a common pin which was driven into the board at the center of the circle. The pin allowed the compass to be moved a little at a time. Two frames were snapped of each position, and dots were added to show the line traced. The effect in the projected picture is of a magic compass moving without the aid of human hands. Of course, the pin occasionally did not hold; consequently, the difficulty of adjusting the compass again in exactly the same position accounted for strange wobbles.

For all our experiments we used a positive film because it gives a high contrast, thus making sharper titles and drawings. Speeds range from 2 to 8, Weston. We preferred the faster film because we could use less illumination; when stand lights were used, we could place them farther from the board. Although all leading manufacturers have good positive film, we used some from Gavaert and Dupont because they have especially good positive film at 8 speed.

As a developer, we used DK 50 because it had sufficient contrast and yet was easy to control. At the time that we made the picture, we were using an "attic" laboratory and "dishpan" developing. Practically everything was experimental, even the developing. The films were wound around a home-made reel 17″ x 18″, which was dipped into three narrow vertical containers for the three processes of developing, fixing, and washing. We mention this to show that one *can* experiment with pictures under conditions of this kind and obtain good results, provided he does not mind a few water spots or occasional flickerings in the finished picture.

As we have shown in this article, we were amateurs, and of course the results were not professional. However, the experience and fun we obtained were well worth the time put into the experiment. In addition, we found the pictures usable and stimulating to our classes. Perhaps more of such experiments by amateurs will help hasten the time when professionals will see that mathematics is a good field for educational pictures.

MAKING AND USING MOTION PICTURES FOR THE TEACHING OF MATHEMATICS

Henry W. Syer

I. Choice of Subject

Educational films are increasing in number daily, but only a few mathematics films are as yet available. The chief reason for this lack is that the proper use of mathematical movies in the classroom is not completely understood. Until more commercial films are available, teachers must make their own. This article is intended to show the kind of films that teachers can make for themselves, and to explain the technique of making them. Many of these films have been completed by the writer; others have been partly finished.

The guiding principle in the choice of subject is that time should not be wasted in making or showing movies when a blackboard demonstration or a set of printed illustrations will tell the story just as well. In all the films listed, motion is an essential part of the argument. It would be very easy to ignore this principle, but such temptation should be put aside. Make films only when motion is an essential part of the argument. For example, one topic on the original list for film production was "Geometry in Art, Nature, and Industry." There was to be a picture of someone taking a pill (with a close-up of the pill) to illustrate the *ellipsoid of revolution,* another of a cook opening a can of peas to illustrate the *cylinder,* and still another of a snowstorm and enlarged snowflakes to illustrate the *hexagon.* When put together all this took too long to give the simple message: "There are many geometric shapes in art, nature, and industry about us every day." Therefore colored projection slides were substituted and these made possible a flexible lecture that could be adjusted to the audience and the time available. The original justification for using *motion* pictures in this situation was the fact that the introduction of *motion* added life and interest and kept attention constant by its novelty.

This aim is important to recognize and to use, but it is not the object of this article to discuss such movies. Their invention is endless—any subject can be dramatized artificially. There is no doubt that a motion picture showing three men (A, B, and C) digging a ditch, or a man rowing up and down a stream, or a swimming pool with water running in and out of it, or piles of money growing larger and smaller would pep up the solution of problems in algebra. But this is too long and artificial a method of presenting a problem in which motion is not part of the mathematical argument. Films for mathematics teaching which inject motion into situations only because motion adds interest are explicitly excluded; most of this can be covered better by field trips, printed illustrations, or collections of mathematical models.

There is another principle that should be kept in mind in choosing a subject for a mathematics movie: *These movies should teach, not merely illustrate.* For example, a picture of a visit to a bank with close-ups of commercial forms is better than never having the class see such forms. But it is usually possible to visit the bank itself or to bring real checks, notes, stocks, and bonds into the classroom. This latter method is much better than using films.

How shall we find suitable subjects if the negative rules above are kept in mind? The teacher who is alert to such phrases as: "if point P approaches P'," "the area approaches a limit," "the number of divisions becomes infinite," "place AB on $A'B'$," "under pressure the circle becomes an ellipse," or "the angle increases" will have no scarcity of subjects. In short, by noting the thousand and one places where *change of position* (motion), *change of shape* (transformation), *correspondence,* or *limit* is implied, topics will be found. The secret is to use motion where it is implied in the mathematical argument, and where the expert and gifted mathematician supplies it with his intuition and imagination.

II. EQUIPMENT THAT IS NEEDED

In each of the following lists the most important equipment needed to produce films for classroom use is catalogued, with the best available placed first on the list. Prices change, and better buys may appear on the market at any time; but the following are suggestions.

A. Cameras (16 mm.)

1. *Eastman Special.* Eastman Kodak Co., Rochester, N. Y. ($417.50). This camera has the advantage of being the most easily adaptable of all the fine 16 mm. cameras, and of being made by a company that is very helpful in solving adaptation problems. There is a definite disadvantage in the high price and in the use of only two different lenses in the double-lens turret. For other uses, where it is important to shift rapidly from one lens to another, this limitation would be a handicap, but not for carefully planned mathematics movies. The high cost of additional equipment is more serious. When investigating this camera, ask about the following features: matched diagonal masks ($11.50), matched quarter masks ($23.00), Model UA motor drive ($185), electric release control ($125), and interval timer ($150). For simple work in a small school with a meager budget these prices may seem high—and they are. They are presented first to tell the whole story of the most adaptable 16 mm. camera sold today. Simpler and less expensive equipment will be found in the other cameras listed.

Do not try to economize by buying and using 8 mm. equipment. The film (being only half as wide) is much less expensive, but it cannot be projected before as large groups as the 16 mm.; the projectors for 8 mm. projectors are not so bright and need darker rooms than those for 16 mm. and less technical equipment is available for increasing the flexibility of the apparatus. For some purposes 8 mm. cameras, film, and projectors are excellent, but not for teaching films.

2. *Bolex H-16.* American Bolex Co., Inc., 155 East 44th St., New York City ($295). Dollar for dollar this is the best buy on the list, but it is placed second because the camera and equipment are made in St. Croix, Switzerland, and there is constantly the danger of having the source of supply and major repairs discontinued. Standard equipment includes automatic threading, three objective turretheads, single frame exposure, hand crank and audible footage counter, and focal plane shutter. To these may be added

synchronous motors (24 frames per second) ($150), Ciné-fader ($17.50), Ciné Transito ($32.50), and behind-the-turret slot for filters ($25).

3. *Filmo 70-F.* Bell and Howell Co., 1801-15 Larchmont Ave., Chicago, Ill. ($243.50). Good foundation camera that gets into the higher price class when the extra features are added. Standard features include three-way turrethead; seven speeds; f 1.5 lens. Possible additions: hand crank ($15), external magazines ($210), electric motor ($85), mask slot device with masks ($160), single frame exposure ($45), wipe-off attachment ($42), rewinding knob ($25), veeder footage counter ($75).
4. *Filmo 70-E.* Bell and Howell ($169.50).
5. *Filmo 141.* Bell and Howell ($160.50).
6. *Filmo 121.* Bell and Howell ($121.50).
7. *Eastman Model K.* Eastman Kodak ($87.50).

Numbers 4–7 on this list are definitely second choice. They can be used, but they require more careful technique in working out the same effects. Each price quoted is for the camera equipped with the more expensive fast lens, which has a greater range of illumination usefulness. If the camera is used only for animated cartoon work with large, expensive floodlights, the prices can be reduced to the following figures by using a slower lens: No. 4—$139; No. 5—$138; No. 6—$91.

B. Film (16 mm.)

(All prices are quoted for 100 feet.)

1. *Cinapro Film.* Sold at local dealers ($1.00 plus $1.00 for developing). This film is adequate for most purposes. Since lighting conditions can be carefully controlled in making the movies, there is no need for fast film; the larger part of the pictures are animated diagrams in black and white and do not require panchromatic or even orthochromatic properties in the film. All that is needed is a film that can be developed for great contrast. This film can be obtained on a white, blue, lavender, or yellow base; and by combining various colored backgrounds the monotony of the type of picture can be lessened.
2. *Eastman Positive Film.* Eastman Kodak Co. ($.90 plus $1.00 for developing). This film is a little less expensive than No. 1, but

being a great deal slower in speed it requires more light or greater camera lens speed. It reverses color values—white becomes black and black becomes white—an advantage sometimes in relieving the monotony or in giving the appearance of animated blackboard drawings.

3. *Agfa Safety Positive.* Agfa Ansco, Binghamton, N. Y. ($1.00 plus $1.00 for developing). Same comments as for No. 2.

4. *Panchromatic Reversible.* Agfa Ansco ($4.50 including developing). A panchromatic film that should be used whenever anything more complicated in color values than black and white drawings is to be photographed, or when it is necessary to have a somewhat faster film.

5. *Panchromatic Reversible.* Eastman Kodak ($6.00 including developing). Same comments as for No. 4.

6. *Hypan Reversible.* Agfa Ansco ($6.00 including developing).

7. *Superpan Reversible.* Agfa Ansco ($7.50 including developing).

8. *Supersensitive Pan. Reversible.* Eastman Kodak ($7.50 including developing).

9. *Super XX Reversal.* Eastman Kodak ($7.50 including developing).

These films (No. 6–9) are used under poor lighting conditions or when the camera is run at extremely high speeds.

10. *Kodachrome, Regular or A.* Eastman Kodak ($9.00 including developing).

Little work has as yet been done with colored teaching films, but there are many possibilities. (See section V-B, p. 342.)

C. Lighting Equipment

There are so many different types of lighting equipment that every movie-maker should consult with his own local dealer.

1. *Photoflood Bulbs.* By overloading the filament, very high candle-power bulbs are obtained which make possible even, fast movies. The smaller size are more economical if the camera has a fast lens, but some films and cameras require the larger size.

2. *Reflectors.* Do not use cardboard reflectors; use metal reflectors with a wide angle rather than the narrow, spotlight type. Do not buy tripods for the lights unless all the work will be done

in one place. Even then, it will be found that the small reflectors with rubber covered clamps can very easily be adjusted onto chair backs and easels. To avoid shadows at least two different light sources should be used.

D. Miscellaneous Material

The following list includes other material which will be needed or might be added to increase the usefulness of the equipment. The list will meet the requirements of a teacher who is making silent movies and showing them to single classes. For example, the projectors are not the best for sound projection or for auditoriums that seat very large crowds.

1. *Projector.* Filmo Master ($139), Film Diplomat ($198), Kodascope G ($135), and Bolex G-16 ($210).
2. *Screen.* For classroom use the best screens are the rear projection screen ($30) or the glass beaded screen (Model B; 30″ x 40″—$9.50). (These quotations are from the Da-lite Screen Co., Chicago, Ill.) If the classroom teacher can obtain a projector with an attached screen that pulls out on extensible arms in front of the lens, he will find it the most useful type of all, because it can be used at a moment's notice in broad daylight.
3. *Titlers.* These are also useful for inexpensive, extremely close-up views.
4. *Editing equipment.* Splicer, "Edicabinet," viewer, cement, reels, and humidor cans.
5. *Masks.* For trick effects.
6. *Extra chambers.* To make it easier to change from one type of film to another.
7. *Wipe-off attachments.* For professional transitions from scene to scene.
8. *Filters.* For controlling light and colors.
9. *Tripod and dolly.* A tripod is necessary for all shots; a dolly is used to follow actors in dramatic shots.
10. *Light meter.* Absolutely necessary. See Weston and General Electric's.
11. *Extra lenses.* For special purposes. Telephoto and wide angle should be the first additions.
12. *Lens extensions.* To make microscopic pictures cheaply.

13. *Electric motors.* To make shots longer than 30 seconds without stops.

14. *Electric time lapse outfit.* For extremely "fast motion" photography. Most useful in biology.

15. *Optical finder.* For advanced work.

16. Drawing board, black and white cardboard, razor blades, scissors, black poster paint, paint brushes, thin wooden and metal strips, thumb-tacks, paper fasteners, black and white thread, etc.

III. TECHNIQUE

There are four useful techniques for homemade mathematics movies: (1) moving charts, (2) cut-out pictures, (3) individual drawings, and (4) direct photography of machines.

A. Moving Charts

Since stop-photography (i.e., using the individual exposure button) is slower and gives poorer results unless the photographer has had a great deal of experience, the most satisfactory way to produce moving drawings is to make cut-out cardboard figures that can be pivoted to turn and change shape while they are being photographed.

"Why," you may ask, "go to the trouble of making large, complicated, and cumbersome animated models and then take motion pictures of them? Why not use the models themselves?" The answer to this question sums up the advantages of movies over many (not all) models for moving illustrations in mathematics teaching. First, films are smaller and therefore easier to transport and store; second, copies can be reproduced more cheaply and in unlimited quantity, by using the same technique of reproduction regardless of the model being reproduced; third, some experiments or illustrations are difficult to operate because the apparatus is temperamental or needs long preparation; fourth, models may be changed in size so that microscopic actions may be shown to many at once or huge machines may b[illegible]ught in[illegible] the classroom to be analyzed part by part; and fi[illegible], scenes ca[illegible] be used that are distant from the classroom or available only certain times of the year. These reasons for producing the films emphasize the supplementary purpose of the films: they are not intended to replace teaching but to enrich it.

The following films will be described as illustrations of the use of moving models:

(1) Line values of trigonometric functions.

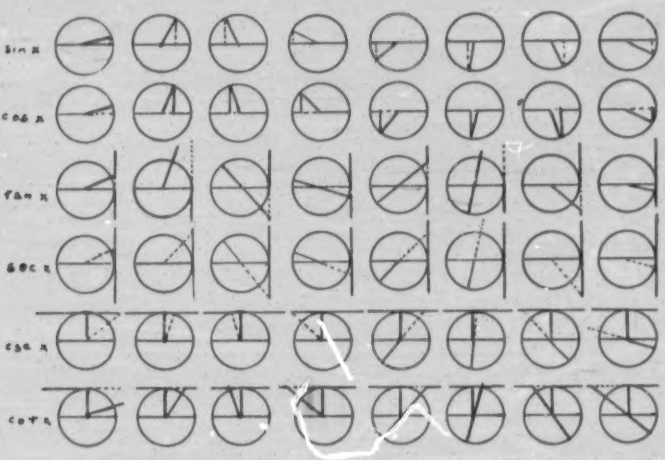
(2) Geometric continuity illustrated by angle-measurement theorems.

(3) Generalized definition of a tangent.

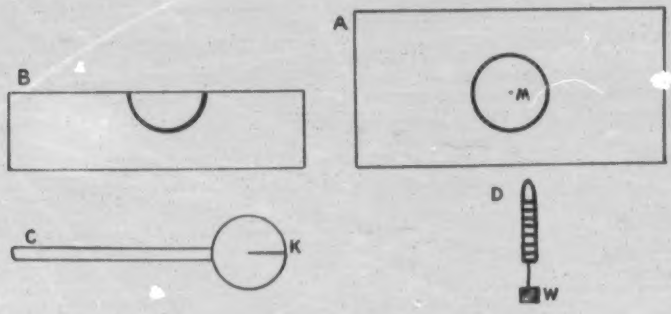
(4) Locus problems; especially cycloid, epicycloid, and hypocycloid.

(1) *Line Values of Trigonometric Functions*

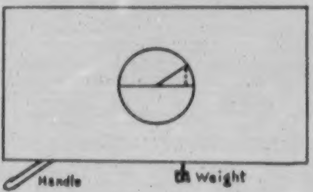
The following six sets of diagrams show the effects desired on the finished film:



(a) *Sin x.* To make the sine of *x* the following cardboard pieces were used:



By placing *C* on *A* so that it pivots at point *M*, and attaching *D* to *C* so that it hangs from point *K*, and then placing *B* over the lower half of the figure, we get the following figure.



Turning the handle from left to right increases the angle positively through the first two quadrants. To make the figure for the third and fourth quadrant, turn *B* over, covering the upper half of the figure; let weight *W* hang over the top of the drawing board to which the cardboard is attached so that it can draw the dotted line upward.

(b) *Cos* x. The upper part of *B* is sliced down one-half inch (at points *R, S,* and *T*) so that a sliding bar *E* can be moved back and forth through the points. Change piece *D* to a solidly colored piece similar in construction. For the first quadrant let piece *E* show through slots *S* and *T*. See Figure 1, above. As the angle increases, decrease the amount of dotted section showing on the front of the diagram to illustrate the cosine decreasing from 1 to 0. For the second quadrant reverse the direction of piece *E* and let it show through slots *R* and *S*. For the third and fourth quadrants turn piece *B* upside down, and run the dotted line back and forth in a similar fashion.

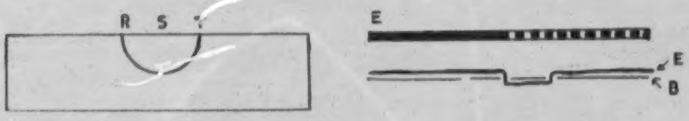


Figure 1

(c) *Sec* x. Make a piece *F* to fit over the right side of *A* to form a tangent to the circle. Extend the initial side of the angle by a long dotted line that will be partly hidden by piece *F*. See Figure 2, page 334. This works for the first and fourth quadrants;

for the second and third quadrants extend the new dotted line across the circle to make a new solidly colored initial side.

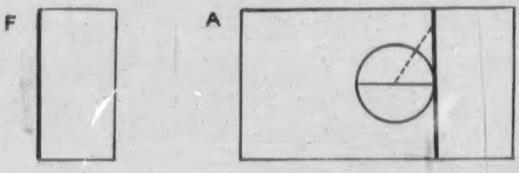
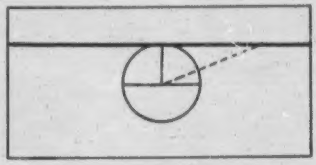


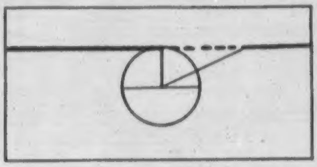
Figure 2

(d) *Tan x*. Similar to the secant.

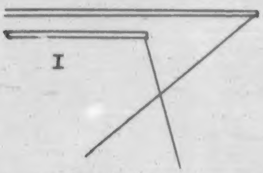
(e) *Csc x*. Make piece *G*, similar to piece *F* in Figure 2, for a horizontal tangent, thus:

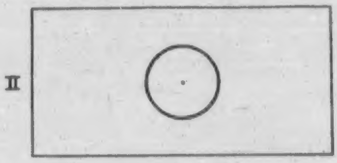


(f) *Cot x*. Similar to the cosecant.

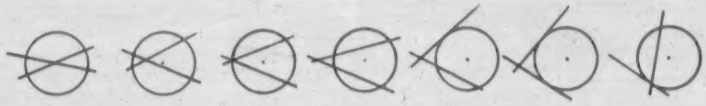


(2) *Geometric Continuity Illustrated by Angle Measurement*



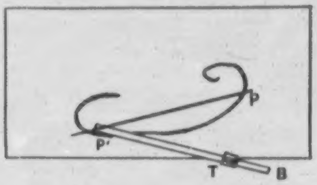


By manipulating the wooden handles of part I, above, in front of part II, we get the following sequence:



This shows that, with proper regard for the sign of the arc length, all these theorems are covered by the single statement: *An angle between two lines cutting a circle is measured by one-half the sum of the intercepted arcs.*

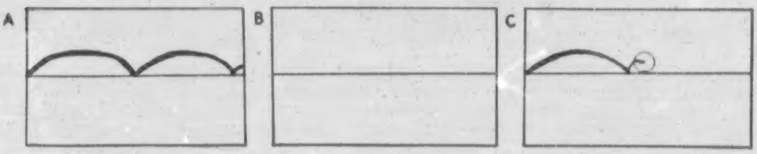
(3) *Generalized Definition of Tangent*



Point T is free to turn and to allow the bar B to slide through it. By keeping the P' end of the bar on the curve, the secant PP' (which rests on a pin projecting upward through bar B) approaches the limiting position of the tangent.

(4) *Locus Problems*

The cycloid, epicycloid, and hypocycloid are fairly easy to show. For the cycloid slide B from left to right across A, placing the circle ⊖ at the correct spot so that the circle appears to trace out the cycloid, as in C.



Similar devices work for other locus problems. Also see section B 2, page 336, for other locus films.

B. *Cut-out Models*

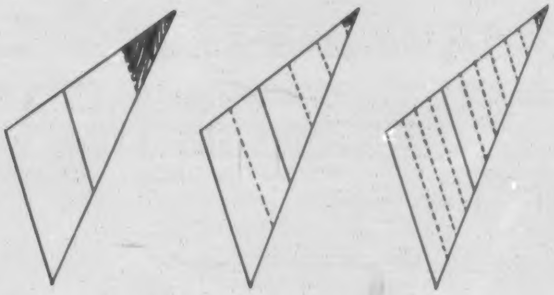
(1) *The Area of a Circle as the Limit of a Polygon*

First make a heavy cardboard circle about 8″ in diameter, and two sets of regular polygons of the right size to be inserted and circumscribed for the given circle, the first set being black and the other set brown. It will be found sufficient to make polygons in each set of 3 to 24, 32, 40, and 48 sides. The following sequences are suggested to be photographed:

I. Inscribed: 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 32, 40, 48
II. Circumscribed: 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 32, 40, 48
III. Inscribed and Circumscribed: 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 32, 40, 48
IV. Inscribed: 3, 6, 12, 24, 48
V. Circumscribed: 3, 6, 12, 24, 48
VI. Inscribed and Circumscribed: 3, 6, 12, 24, 48
VII. Inscribed: 4, 8, 16, 32
VIII. Circumscribed: 4, 8, 16, 32
IX. Inscribed and Circumscribed: 4, 8, 16, 32
X. Inscribed: 5, 10, 20, 40
XI. Circumscribed: 5, 10, 20, 40
XII. Inscribed and Circumscribed: 5, 10, 20, 40

(2) *Line Parallel to One Side of a Triangle*

To illustrate the theorem *A line parallel to one side of a triangle divides the other two sides into proportional segments* when the "other two sides" are incommensurable, the following series of cutouts is used:



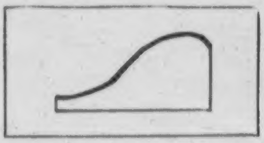
The darkened part of the triangle shows the amount of inaccuracy in the approximation; the inaccuracy decreases as the unit of measurement is made smaller.

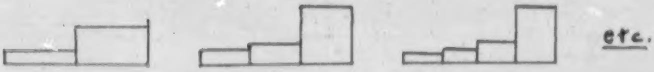
(3) *Cones, Cylinders, and Spheres as Limits*

By making plaster models of cones, cylinders, and spheres and by gluing together sheets of transparent plastics to make pyramids, prisms, and polyhedra which will fit over the plaster models, and then proceeding as in the pictures of polygons circumscribed about a circle (B 1, page 335), we approach the solid figures as limits of circumscribed polyhedra. Reversing the process and making cones, cylinders, and spheres of transparent plastic material, and plaster models of inscribed pyramids, prisms, and polyhedra, we get the curvilinear solids as limits of inscribed polyhedra.

(4) *Area under a Curve*

I





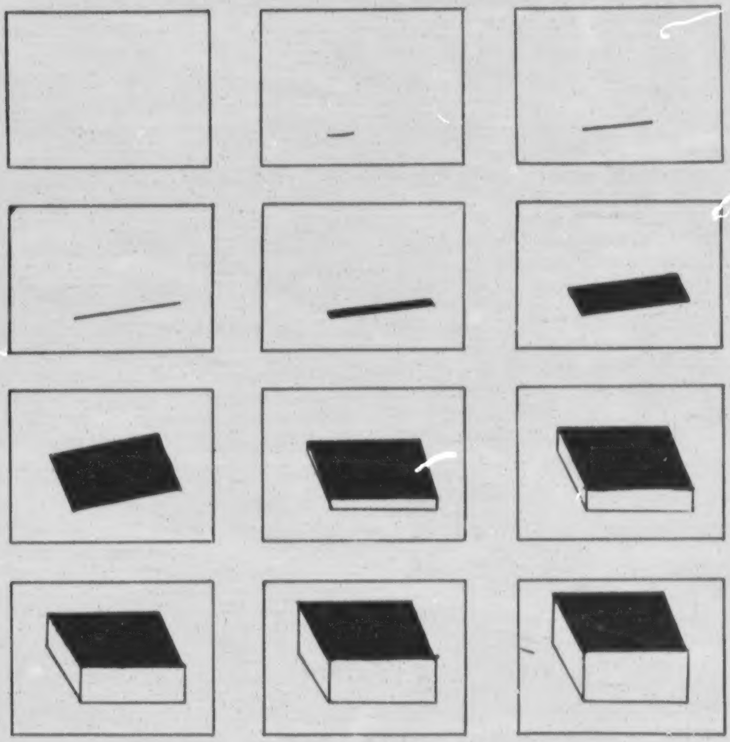
On Figure I place the series of cut-outs shown above, to illustrate the fundamental reasoning of the integral calculus. Be sure to allow enough time in showing each figure so that it can be comprehended. Do not try to make the picture a gliding progression from one area to the next. The same type of selections should be photographed as suggested in B 1, page 335.

C. *Individual Drawings*

If the large amount of change in a diagram makes moving models impossible, the standard Hollywood technique of making many similar drawings and photographing them consecutively to give the effect of motion may be used.

(1) *Dimensions—Point to Line to Plane to Solid*

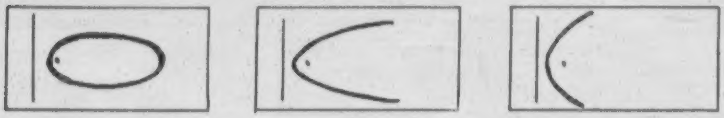
This film is intended for elementary geometry to show the three-dimensionality of geometry, or as a beginning of a discussion of non-Euclidean geometry, or geometry of more than three dimensions. The sequence looks something like this:



(2) *Conic Sections—Ratio Definitions*

Start with a point and a line thus:

Find the locus of a point moving so that the ratio of its distance from the point to its distance to the line is ½, 1, and 2, respectively; getting the following results:



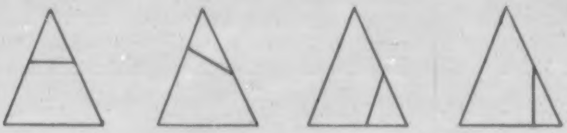
(3) *Wave Forms*

Arrange cut-out letters on a sheet of cardboard with faint vertical and horizontal lines for guidance. By moving the letters up and down, transverse wave motion is shown; by moving them sideways, longitudinal wave motion is shown.

D. Direct Photography

(1) *Conic Sections—Clay Cones*

With a knife cut four clay cones in this manner:



and show the results as the four conic sections.

(2) *Conic Sections—Flashlight*

The conic sections are illustrated by shining a photoflood bulb with a reflector on the wall and taking movies of the pattern as it changes from one angle to another.

(3) *Conic Sections—Mechanical Construction*

Actual scenes of the string, T-square, and triangle constructions for the circle, ellipse, parabola, and hyperbola are easy to photograph.

(4) *Trigonometric Functions—Applications of Trigonometry*

Take photographs of the oscilloscope analyzing sound waves and of the action of pendulums (simple and compound) tracing out curves.

(5) *Statistical Concepts*

"Distribution" is illustrated by having, for example, 75 boys who are lined up in alphabetical order file into a formation behind signs which state their heights (5′9″, 5′10″, etc.). The biggest group steps forward to illustrate "mode." By pairing the tallest with the shortest successively along the line the meaning of "median" can be seen. Other ideas from statistical terms follow easily.

E. Colored Slides

A series of projection slides illustrating "Geometric Shapes in Art, Nature, and Industry" on Kodachrome film has been started. Although this topic was first treated in a motion-picture film, it was found that teachers place the emphasis on a different place in the subject and so resent the automatic control of the timing that a film demands. As more and more examples are collected, the presentation of the subject can be adapted to the time of year, and to the age and interests of the class much better than if a standardized film is used. The chief difference between this and the preceding topics is that the stereopticon slides are intended to increase appreciation of mathematics, the films to increase understanding and insight.

The solid shapes—triangular prisms (both equilateral and right angled), square prisms (parallelepiped, square plinth, and cube), hexagonal and octagonal prisms, triangular and square pyramids, complete spheres and hemispheres, regular ellipsoids and ovoids, cylinders and circular plinths, and cones; and the plane figures—circle and its parts, semicircle, sector, segment, arc and chord, the ellipse, the oval, the parabola, and hyperbola, and finally the polygons, the triangle (right-angled scalene, right-angled isosceles, acute-angled isosceles with no right angle, obtuse-angled isosceles, obtuse-angled scalene), the quadrilateral (rectangle, square, and rhombus), the pentagon, the hexagon, and the octagon—are illustrated by hundreds of examples of shapes in the world about us. Think of seeing elliptic cogwheels in a machine, enlargements of hexagonal snowflakes, square prismatic crystals of pyrite being mined, an ovoid disappearing into a cake as the familiar egg, the spiral pattern of sunflower seeds on the plant, or of sea shells, the parabolic flight of a tennis ball, or cartoons of the planets moving in elliptic orbits—all in colored pictures. Many other examples have been collected from questionnaires sent out to schools.

IV. TOPICS FOR FILMS

1. Geometric Continuity
    A. Angle Measurement in a Circle (see III A 2, p. 324)
2. Limit Theorems in Geometry
    A. Area of Circles as Limit of Inscribed Polygons (see III B 1)

- B. Area of Circle as Limit of Circumscribed Polygons (see III B 1)
- C. Area of Circle as Limit of Inscribed and Circumscribed Polygons (see III B 1)
- D. "A line parallel to one side of a triangle divides the other two sides proportionally" (see III B 2)
- E. Cone as Limit of Pyramids (see III B 3)
- F. Cylinder as Limit of Prisms (see III B 3)
- G. Sphere as Limit of Polyhedra (see III B 3)
- H. Generalized Definition of Tangent—Spiral (see III A 3)
- I. Generalized Definition of Tangent—Limit of Increment Quotient
- J. Area under a Curve (see III B 4)

3. The Conic Sections
    - A. Cutting of a Clay Cone (see III D 1)
    - B. Flashlight on the Wall (see III D 2)
    - C. Ratio Definitions (see III C 2)
    - D. Mechanical Constructions (see III D 3)
    - E. Surfaces of Revolution
    - F. Effects of Constants on the Graph's Position
4. Wave Forms
    - A. Longitudinal Waves (see III C 3)
    - B. Transverse Waves
    - C. Combinations of Waves
    - D. Harmonic Motion and Resolution of Wave Motion
5. Trigonometric Function
    - A. Line Values (see III A 1)
    - B. Graphs
    - C. Applications (see III D 4)
6. Statistical Concepts
    - A. Distribution (see III D 5)
    - B. Mode (see III D 5)
    - C. Median (see III D 5)
7. Locus
    - A. Cycloid, Epicycloid, and Hypocycloid (see III A 4)
    - B. Conic Sections (see III C 2)
8. Dimensions
    - A. One-dimension Land
    - B. Two-dimension Land
    - C. Three-dimension Land
    - D. Four-dimension Land
    - E. Point to Line to Plane to Solid (see III C 1)
9. Non-Euclidean Geometry
10. Graphing Complex Numbers
11. Statics and Dynamics
12. Nomographic Charts

13. Map Projections
14. Principle of Duality
15. Symmetry (Axes and Points)
16. Transformations of Coordinates

V. PROPHECIES FOR THE FUTURE

(A) Developments in Sound

Just as every illustration in a book should have its title, so each of these films should have its own sound track carrying the simplest and shortest explanation. Just as we often look at an illustration and see more than is called to our attention by the sub-title, so these films should be shown again and again with the sound track disconnected, and with the instructor giving newer and deeper interpretations of the illustrations in keeping with the increasing mathematical maturity of the class.

Further developments in sound may lead to increased interpretation of the figure by the accompanying sound. For example, as a figure increases in size, the volume of the voice describing the action might increase with it to suggest its growth. Again, if two figures are being compared alternately, it will be helpful to have a man's voice and a woman's voice give the alternate descriptions successively to further contrast the properties.

A musical accompaniment to a dynamic mathematical film may serve to strengthen the relationships between its parts and properties, but there is so much room for argument over the proper music for interpreting geometric relations that such experiments will be completely accepted only by those who make them. Nevertheless, I should like to see made a mathematical movie that has its appropriate score written to develop in music the relationships that are shown in lines and planes. There have been some attempts to use the "prettiness" of geometric forms to accompany standard musical compositions, but this present appeal is for music which shall be subordinate to the mathematics of the figures and yet shall strengthen them.

(B) Developments in Color

Every mathematics teacher knows the advantage of tracing in color some part of a figure to make it stand out as a unit without destroying its relationship to the rest of the diagram. Imagine the advantages of having the colored lines appear, disappear, and change color instantly without interfering hands, and as quickly as the description of the figure proceeds, with as little effort as drawing a point on the blackboard. Again, imagine the advantages of a rainbow of colors fading into one another to show changing relations; allowing types of lines, direction of motion, description by voice, description by printed labels, and color to supplement each other in the description. The greatest danger in the use of color is that it will be used to excess for the purely beautiful effects of colored, movable geometric figures. Beauty in geometry is desirable until it attracts so much attention that it detracts from the teaching of facts and relations that the film set out to do.

(C) Three-dimensional Movies

There are two chief ways of making three-dimensional movies at the present time: the color-filter method and the polarized light method. In the first, a red picture and a blue picture of the same subject are made on the same film. These views are taken with two lenses as far apart as a person's two eyes, or slightly farther. Whoever views the movies looks through colored glasses that allow each eye to see only one of the images on the screen, thereby giving the impression of three-dimensions. The other method uses polarized light to supply each eye with the two different images. The two pictures come from the screen on light waves polarized in perpendicular planes, and this time the glasses that the viewer uses are Polaroid, which filter out the pictures in the same manner as the color-filters did before.

All three-dimensional movies made thus far have been straightforward presentations of three-dimensional objects behaving in the usual way. It would be helpful to make a series of films showing the figures of solid geometry building themselves up from disintegrated points, lines, and planes, or showing figures taking themselves apart. For example, a triangular prism could separate itself into three triangular pyramids, a hexagonal pyramid could separate itself into four triangular pyramids, or the limit theorems (see III B 3) could be demonstrated beautifully in three-dimensional films.

In addition to true-to-life demonstrations of solid geometry, it would be interesting to make greater use of the peculiar advantages of moving pictures over ordinary models. In plane geometry films we used figures which changed shape, position, and color without distracting pauses or outside aid. This continuous and swift succession of illustrations is fast enough to keep up with a spoken description, or even as fast as the thought processes that are developing the idea. Thus no time is lost erasing pictures from the blackboard, changing lantern slides, or holding up illustrations, because the illustrations and thought move simultaneously. Keep these advantages in three-dimensional films and also let them develop imagination in their own medium. For example, try to suggest four-dimensional figures (such as the hyper-cube) by moving three-dimensional figures; or use movies to show intersections of solids moving right before your eyes—the solids passing through one another and the intersection standing out brightly like a halo.

VI. Use in the Classroom

(A) Not a Substitute for a Proof

The teacher should not expect the film to do the teaching for him. *The picture is only an illustration,* and should be used exactly in the same way as a picture in a book or a wooden model held up before the class; the only new feature is its property of motion. Since the fundamental premise is not to waste time on films when motion is not an essential part of the argument, we have in these pictures an entirely new type of illustrative material which will not replace teaching but will broaden and supplement it.

The same film can be used for many purposes: introduction of a subject, explanation, review, or testing. For example, in discussing line-values the teacher would say, "If we let the radius rotate, you can see the line-value of the sine increase to 1, then decrease through 0 to -1, and finally again increase to 0. Let me show you what I mean." Then the film (taking about two minutes) is run off and the explanation proceeds with no more interruption than the holding up of a picture or the drawing of a diagram. Again, as review this time, the instructor might say, "You remember how the line-values of the six functions look," and then run off all six parts (about 8 minutes) as a quick review. Again, the parts can be used as a test, the class being asked to write the values of the functions of 0°, 90°, 180°, 270°, 360° as they occur for each function.

Each film illustration should be presented with as little detailed ceremony as possible. There should be no appearance of a show or entertainment; it is just another, new way of getting across a point that is difficult for some to see. Ideally each classroom should have a permanent projector and screen and library of the films that are to be used more than once. It is too much to expect a teacher to send away for films for teaching weeks ahead and then not to look upon their use as something of a novelty. To speed up the actual projection during the period, a student should be trained in the use of the projector so that the teacher can ask for a certain film, have it shown, and then go on with the class with little interruption. Naturally this ideal plan must be modified as the budget demands, but it should remain the ultimate goal in planning for the classroom use of films.

(B) Not Try to Tell a Story

Too often films are made pretty by using Hollywood techniques of closeups, symbolic pictures of gears, hands holding money, or meaningless rotating triangles where these have no place. Do not attempt to replace all former mathematical illustrations with motion pictures. Unless it is impossible to do so, movies should not replace actual models in solid geometry, real trips to the bank to see how our financial system works, actual views through a telescope to see planets changing their positions in their orbits, the oscilloscope and pendulum as harmonic motion, patterns of snowflakes through a microscope, or laws of the lever and similar illustrations. By all means use movies for these realizations when time must be economized or when the actual objects are too expensive, too large, or too far away to see. Let them fulfill their own new and peculiar service—of contributing motion to illustrations that tried to suggest ideas of motion but actually lay dead on the pages of a book. Develop this new type of illustrative material and extend the methods of teaching beyond their present boundaries.

GEOMETRIC STEREOGRAMS AND HOW TO MAKE THEM

Walter F. Shenton

THE COINED expression "geometric stereograms" is used as a name for plane drawings of a space figure produced in two colors and so executed that when viewed through a pair of bi-colored viewing glasses, they give a vivid presentation of a real space figure.

It is rather common knowledge—subject to very simple checking—that the appearance of a near-by object in three-dimensional space depends on the fact that each of our two eyes sees a different picture of this object. One can easily verify this fact by setting a pin vertically in a drawing board and marking on the board the places where the pinhead seems to be when viewed by the right eye with the left eye closed, and vice versa. The aggregate of points, lines, and planes in a space figure forms with the optical center of each eye a rather complex pencil of rays. The plane figure formed by the lines and points of the intersection of such a pencil with any plane, not too greatly inclined to the general direction of the pencil, will replace the entire pencil formed by the space figure with respect to the viewing eye. If we can now contrive to get on a single background plane the sections of the two pencils of rays formed at the optical centers of our two eyes and so regulate the drawings that each eye shall see only the section which *belongs* to it, the effect will really be the same on the eyes as that caused by viewing the original solid figure. Of the many methods of accomplishing this separation, one of the oldest and simplest produces such excellent results when used to reproduce geometric figures in space that we explain it rather fully in this article. For the readers who may care to follow the historical development of this type of stereoscopic representation reference is made below to a previous article by the author [1].

If you look through a red glass at a red drawing on a white sheet, the red figure will completely disappear in a red background. This same figure, when viewed through a blue glass, will appear as a vivid black drawing on a blue ground. Similarly, a blue drawing (the shade will have to be chosen carefully to match the blue glass) will disappear when viewed through the blue glass, but will appear densely black through the red glass. A plain white sheet, when viewed through the two almost complementary colored glasses, one over each eye, will assume a light neutral color. This simple color principle makes it possible to draw the sections of the point-projections of a space figure from each of the two eyes in a color complementary to the glass used over that eye for viewing the stereogram. As a result, the eyes accept a sharp black space figure instead of the conglomerate red and blue drawing.

For the past thirty years the author has been drawing three-dimensional figures by using some very simple applications of ordinary orthographic projections. In the past five years he has been working on the idea that these figures could be produced by high school students in connection with their study of solid geometry. Recently a number of tests have been made with such students, and in each instance, after less than an hour's instruction, satisfactory figures have been drawn to the accompaniment of the greatest enthusiasm of the young artists themselves.

In order to make these methods available to other teachers and students we shall present them here in this order: (1) a résumé of the principles of orthographic projection—with apologies to the teachers of mechanical drawing; (2) the solid geometry validation of the methods used in making the stereograms; and (3) a simple but detailed explanation of the methods actually used in making the pictures, including the layout of the drawing board and the colors necessary for the best results.

A RESUME OF THE PRINCIPLES OF ORTHOGRAPHIC PROJECTION

The orthographic projections of a solid consist of the three projections of that solid on three mutually perpendicular planes.* The common point of the three planes is called the *origin, O;* the three mutually perpendicular lines in which the planes intersect in pairs are the *coördinate axes, OX, OY, OZ.* The view which is the projection of the solid on its base-plane, designated as P or T, or on any plane parallel to it, is called the *plan* or *top view;* the projection on the plane between the observer and the solid is called the *front view* (we shall call its plane F); the projection on the other plane, called S, is the *side view.* The actual relationships of this method of projection are shown in an isometric drawing of a regular tetrahedron and its three orthographic projections (Fig. 1). The plane XOY is the "top" of the "box" on which the tetrahedron is projected. Plane YOZ lies between the observer and the object, and plane XOZ is to the left of the observer. To save confusion in the figure, the projecting planes of the single point B have been drawn in plainly, while the other projecting planes are only faintly marked. The plane $B\,B_T\,B_X\,B_S$ passes through B perpendicular to the planes XOY and XOZ, parallel to YOZ, therefore perpendicular to the line OX; $B\,B_T\,B_Y\,B_F$ is similarly perpendicular to XOY and YOZ, parallel to XOZ, therefore perpendicular to OY; while $B\,B_F\,B_Z\,B_S$ is perpendicular to XOZ and YOZ, parallel to XOY, therefore perpendicular to OZ. It will be noted that *each* of these planes contains the actual point B of the solid and the projections of B on two of the coordinate planes and that the six traces of these planes (shown in dot-dash

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* By projection we mean the ordinary perpendicular projection of plane and solid geometry.

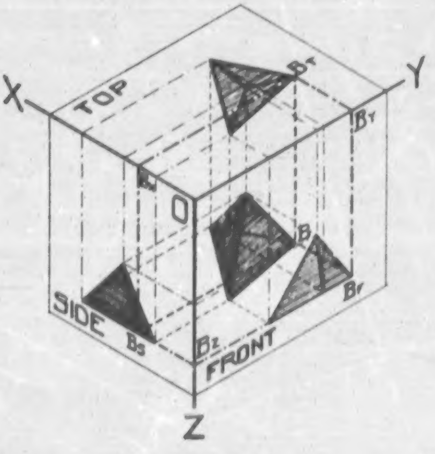

Figure 1

lines) in the coordinate planes are perpendicular to the coordinate axes in pairs.

If now we were to dispose of the tetrahedron, cut this figure along the line *OY* and lay it out in one plane as in Figure 2, we would have the layout of an ordinary orthographic projection in a single plane. Of course, the line *OY* which we have cut apart will occupy two positions 90° apart; but if we join the intersections on one of these *OY* lines with those on the other by means of quadrants of circles with their centers at *O*, we show how the three sections are related when drawn in one plane.

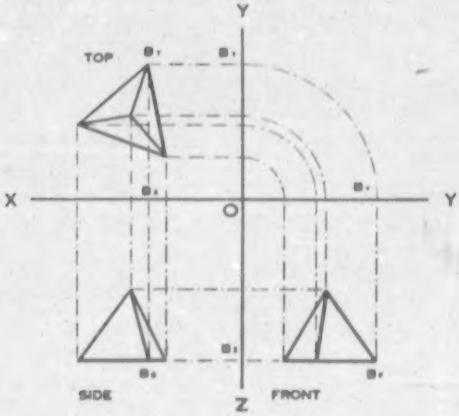

Figure 2

Let us suppose, now, that we wish to make these projections without the intermediary of such a space-drawing as we have in Figure 1. For simplicity, let us place one base of the tetrahedron in a plane parallel to *XOY;* the projection in that plane will then be the undistorted view of a single face of the tetrahedron together with the three lines which join the vertices of this equilateral triangle with its centroid. The front view and side view will then have the base plane represented as a line parallel to *OX* and located at a convenient height; the vertex will be located on a line parallel to this base line and at a distance equal to the altitude of the tetrahedron above it. (The teacher will appreciate the motivation which this *needed* computation will offer to the student.) Let us now trace the route of the projection of a single point B_T in the top view. Its projections will lie on $B_T\,B_Y\,B_F$, on $B_T\,B_X\,B_S$, and on $B_S\,B_Z\,B_F$ and the projections in the side and front views will be $B_S\,B_F$ respectively. The other three points are similarly projected by means of lines perpendicular to the three axes; their projections and the lines joining them form the side and front views. You will note that these lines correspond to the similarly marked dot-dash lines in Figure 1. We have arranged our projections so that the top view and side view are placed vertically on the sheet, with the top view above the side view. This plan conserves space in making our commonest stereoscopic projections, as we shall see later.

Drawing a few of these orthographic projections to illustrate the earlier theorems in solid geometry will, in itself, foster a correct idea of the meaning of the space relations and will shortly drive the student into voluntary conference with his teacher of mathematics or mechanical drawing to find out how to make various oblique sections of his figures and similar problems of descriptive geometry.

METHODS OF DRAWING VALIDATED BY SOLID GEOMETRY

Now that we have learned to make the top and side views of a figure, let us examine and validate the projections necessary to make its stereograms. In Figure 3 we have a sketch of a set of three planes used to explain this projection. It shows how we make a stereogram of a figure which is to appear as though it were a model standing on our desk. In planes S and P you will see the side view and plan (*plan* is an easier term to use than *top view*) of a square-based right pyramid $ABCD$-V. The observer is supposed to be behind the plane to the right, with his eyes actually in the posititions R (right) and L (left) in a line ERL parallel to plane P—and hence to line OX—and perpendicular to plane S at E. He would then look down on the projections or on the "model." If we could join the points R and L with the point V of the actual model, the two lines thus obtained would cut the plane P in the two points V_R and V_L which are the desired projections of V for the two eyes.

Obviously it is too much to ask that we have an actual physical model for each picture we wish to make, so we shall have to interpret this space projection in the light of a corresponding plane projection of some sort. The line which joins V with its projection V' in P is obviously perpendicular to plane P; so also are the lines which join L and R to their projections L' and R'. These three lines are then parallel to each other. A single plane will pass through R and V and their projections on P, and it will be perpendicular to P so that its projection on P will be its trace $R'V'$ in that plane, and V_R will lie somewhere on $R'V'$; similarly V_L will lie somewhere on $L'V'$. Again, the lines ERL and VV'', being perpendicular to plane S, are parallel; hence a plane passing through R, L, and V would cut plane S in the straight line EV'' and the plane P in V_PF. V_PF is perpendicular to OY at the point V_P which is the projection of V'' from E on OY in the plane S. The two projections V_L and V_R must lie in this plane; hence they must be located where the trace of this plane in P meets the two lines $R'V'$ and $L'V'$. This proves that the projections of any point V,

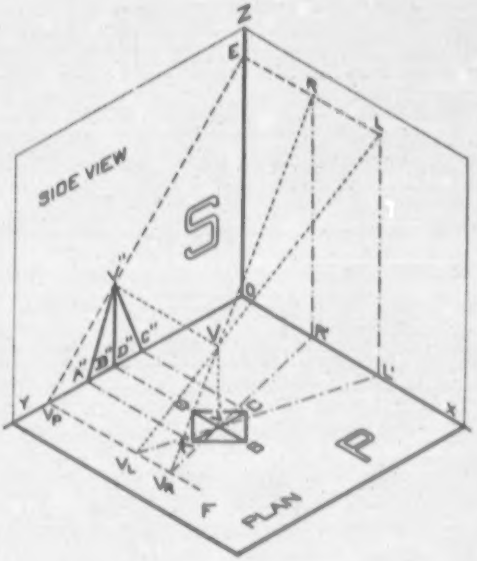

Figure 3

not in plane P, from the two eye positions L and R onto the plane P may be found by (1) drawing lines $R'V'$ and $L'V'$ in plane P or (2) finding where these two lines are cut by a line perpendicular to OY through the point V_P where EV'' cuts OY. All points which lie *in* the plane P will be projected into themselves; that is, A, B. C, and D will remain A, B, C, and D under the projection. If now the points A, B, C, D, and V_R are properly joined with a blue pencil, and A, B, C, D, and V_L are similarly joined in red, we have made a stereogram of the square-based right pyramid which, when viewed through a pair of glasses, red over the right eye and blue over the left, will give a single black pyramid in space when the spectacled eyes are in positions R and L. This is the type of projection which Vuibert [2] calls horizontal and marks H in his beautiful brochure.

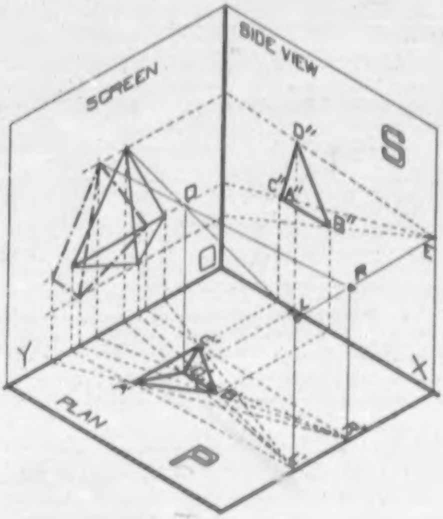

Figure 4

The other projection is that which is made against a vertical background, such as would be needed if the pictures are to be thrown onto a screen and viewed by a class. Figure 4 is a sketch of the three planes set up for the demonstration of the validity of this projection. In this model the actual positions of L and R, the eyes of the observer, are shown vertically above the marks *R'* and *L'* and in a line passing through *E* perpendicular to the plane *S* and therefore parallel to the plane *P*. The discussion here differs from that of the previous paragraph only to the extent that the plane on which we are projecting is *not* one of the planes of the orthographic projection and the traces of the projecting planes are therefore a network of lines at right angles to each other in the plane of the screen which is perpendicular to the planes containing the plan and the side view. The figure used for demonstration is that of a regular tetrahedron and the projection for the right eye is shown in dot-dash lines, while that for the left eye is shown in solid lines. In this projection the observer is facing the screen and the image will usually lie between him and the screen.

Practical Details for Drawing Geometric Stereograms

Before we give the actual dimensions and layouts which have been found convenient in making stereograms, let us face a few practical facts. The blue and red glasses of which we speak are made by dyeing the gelatine film on lantern-slide plates after the silver has been removed. The dyes used are those of the Eastman Kodak Company marketed under the names "cyan A" and "magenta B." These dyes are also used in making transparencies for lantern slides and imbibition prints of stereograms. Those interested in this phase of the work are referred to the proper Kodak manual [3]. A set of pictures for fifteen solid geometry theorems have been gathered together by Dr. Breslich of the University of Chicago. They are furnished with viewing glasses called "Orthocopes." Extra glasses are advertised for sale by the publishers [4]. These glasses reverse the colors we have used and that fact must be taken into account in making the drawings. Making stereograms presupposes the availability of a drawing board, T-square, triangles, and thumbtacks or Scotch drafting tape. Faber's Mongol Red 866 and Mongol Blue 945 have been found satisfactory pencils for making the drawings.

The projection we have found most interesting for student use is the one in which the figure is viewed as though it stands in or is suspended above a horizontal plane; that is, as though it were a model standing on a desk or suspended above the desk. Figure 5 shows a convenient layout for the upper right-hand corner of a drawing board laid out for drawing on ordinary 8½" x 11" typewriter paper, which is less expensive than drawing paper and serves almost as well. Along the right-hand side about a half inch in from the edge is the *eye line*. At the very top of the board, drive in a small brad or a phonograph needle at the point marked *E*. About 7 inches further down on this line, drive in another brad for *R* and at a pupil-distance (about 2.4 inches) another for *L*. At about 15 inches below *E* draw a horizontal *base line* across the board. A line parallel to *ERL* and about 8 inches to its left makes a good guide line for the front of the drawings. The sheet of paper may now be placed in a position about an inch over the guide line and about an inch lower than the base line. The T-square should be long enough to cover the sheet, but it will be more convenient if it is short enough to clear all the brads. With the board laid out in this fashion we commence to draw a stereogram.

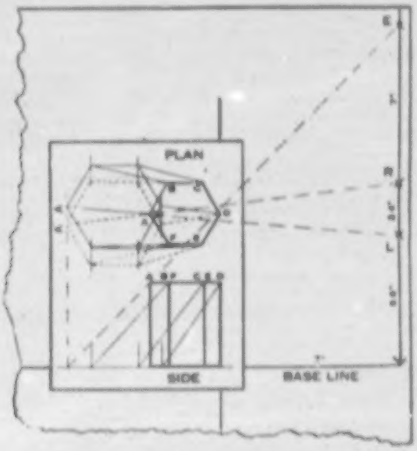

Figure 5

In Figure 5 we have chosen to present a regular hexagonal right prism and have placed the plan with its front points on or near the vertical guide line. The side view shows six points, *A, B, C, D, E,* and *F* to be projected. Using a straightedge—the T-square turned over will serve—pressed against *L* as a pivot, draw lines toward the left from each of the points of the plan which are to be projected, like the solid line *AA*. Repeat, using *R* for a pivot, getting lines like the dotted *AA*. Now use *E* for a pivot and project the similarly marked points in the side view onto the base line. Draw vertical lines from these intersections with the base line, like the dot-dash *AA*, and the intersections of the three lines determined by the same point will give the two projections for that point. Proper joinings of these points will complete the stereogram. In this sketch, the right-eye figure is dotted in and the left-eye figure is solid. If you wish to have your stereogram lettered, you will have to project the little rectangles in which you wish to have the letters—a tedious but not impossible job. In this type of projection, the figure in the base plane forms a part of each projection, and should therefore be traced with both colors. If you follow these directions and then color your two figures to match the glasses, you will see a fine right prism when you place your diagram on your desk about eight inches from the edge and fifteen inches below your eyes. These dimensions are somewhat arbitrary and may be changed to suit your convenience, but they will produce nice figures. We recommend pricking the intersections through on a clean sheet of paper or card and putting the colored lines on this sheet with a sheet of press board or a sheet of plate glass underneath to avoid the depressions which pencils will make if a softer back is used. These depressions will show as shadow traces even if the colors and glasses are perfectly matched.

For those readers who wish to make charts for vertical viewing, we present a layout for this method in Figure 6. To check this figure, you will have to move the *eye line* to the right twice the length of the *base line,* for it was impossible to include it in the figure and get any detail in the drawing. The eye line should be from twelve to eighteen inches away from the *screen line.* The figure projected is that of a regular tetrahedron with one edge in the screen and with the base horizontal, viewed from a point slightly above the figure. The side view is arranged above the plan for this projection and all the projections with *E, L,* and *R* as pivots are made on the screen line and the measurements from the projections of the points in the side view are transferred around one quadrant so as to be on the right edge of the screen. Horizontal and vertical lines are drawn from these projection marks on the two edges of the screen, and the intersections of corresponding lines give the points of the stereogram. The line *BC* is a part of each colored figure with *A′* and *D′* the two projected points for the right eye and *A* and *D* for the left eye. Of course, this figure will have to be set up vertically to appear properly.

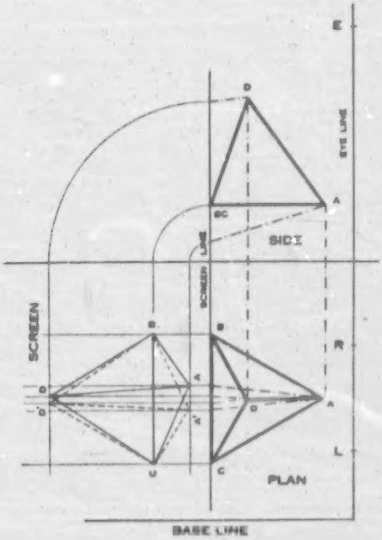

Figure 6

It would be most unfair to the readers not to warn them that when they start to draw geometric stereograms, they are entering into a new form of descriptive art. Just as we turn a real space model into various positions to obtain the best view of it, we must be content to start a more or less complicated figure many times to obtain the most favorable result. It is the author's plan shortly to present a portfolio of stereograms of the most needed figures in solid geometry, together with an amplification of these directions and a series of problems to be projected. The sponsors of this yearbook will be informed when, where, and if it appears.

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4. **BRESLICH**, E. R. *Diagrams in Three Dimensions*. Newson and Co., New York. (N.D.)

THE STEREOGRAPH AS A TEACHING TOOL*

John T. Rule

THOUGH THE POWER of the three-dimensional photograph as a teaching tool has long been known, its full utilization has awaited an adequate method of projection. Such a method became available with the development of inexpensive materials for polarizing light. The full possibilities of the medium, however, are not yet realized by many teachers who will be eager to use stereographs when they realize that the three-dimensional picture is a great deal more than a flat picture with a pleasing sense of depth added.

It is important, therefore, to review the chief advantages of such pictures.

First, the flat picture is fixed only in direction, whereas its three-dimensional counterpart is fixed in size, shape, and direction. It is therefore measurable in three dimensions. The dimension of depth is real and visually obvious, and need not be inferred from perspective. This is a great advantage in teaching such subjects as solid geometry, analytical geometry, astronomy, navigation, and crystal and atomic structure. The problem of seeing a flat drawing or picture solidly is immediately eliminated. Three-dimensional drawings for solid geometry and photographs of machines and machine parts for practical mechanics illustrate this advantage. Furthermore, the accurate measurement of depth is absolutely essential in such sciences as aerial mapping.

Second, the stereoscopic picture has much greater separation power than the flat picture. In the ordinary photograph of complicated apparatus with many individual parts partially obscuring each other at different depths, it frequently is impossible to distinguish between the parts. The three-dimensional counterpart of the same picture will make the entire pattern clear. I think of two good examples of this quality. One is a stereoscopic drawing of an icosahedron. From the flat picture the student is unable to count the number of faces, but he can do so from the solid picture. The other is a picture taken in a chemical laboratory where it is important to know the number of glass tubes used in an experiment. The number of tubes can be easily counted in the solid picture and not at all in the flat one.

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* Reprinted by permission of *Visual Review*, 1942.

Third, the psychological appeal of the stereograph is initially tremendous and has an unusual persistence. It captures the interest of everyone. There is an aesthetic satisfaction in the quality of depth which is difficult to explain but nevertheless exercises an enormous power over attention. This is a very real and important phenomenon. Students will look at three-dimensional pictures long after their interest would otherwise have flagged. Furthermore, with each new experience this interest will return with its original freshness.

This quality is of the utmost importance in teaching such general subjects as geography and history. The depth interest of stereographs makes the subject matter fresh and real. This is why stereographs of trips around the world have always had a good market.

These advantages together with adequate means of presentation will make the stereograph extremely effective and popular as a teaching tool.

It should be mentioned that, scientifically, the three-dimensional photograph has already come into its own, being standard wherever depth measurements are made. Thus in aerial photography, range finding, map making, and X ray, many complicated machines are in use for interpreting and measuring depths in three-dimensional photographs.

Certainly the teacher who has not investigated this field may do so to his advantage.

AVAILABLE EQUIPMENT AND PRINCIPLES OF USE OF VISUAL AIDS

M. Richard Dickter

THE PURPOSE of this article is to indicate briefly the various types of visual aids in current use, the apparatus necessary for using each one, and a working technique for teaching with visual aids. It will concern itself with visual aids that are comparatively new to mathematics in the sense that few of them deal directly with this subject. The development of content material for each of these several aids offers a new and valuable field of endeavor to teachers of mathematics. Once the content is established—and appropriate content for each of three purposes is needed: to motivate interest in mathematics, to enrich the subject matter of mathematics, and to provide specific instructional materials—it becomes necessary to decide what type of visual aid is best adapted to the particular material. This description of available visual aids will be helpful in clarifying the problem of choice.

OPAQUE PICTORIAL MATERIALS

Description. Opaque pictorial materials, such as photographs and illustrations on post cards, from books, magazines, and similar sources, are numerous, easily obtained, and relatively inexpensive. A single picture is frequently more effective than a lengthy description. The main limitation of these pictures is that they can be seen by only one student, or a small group of students, at one time, unless there is a sufficient number for the whole class. However, this difficulty can be overcome by using an opaque projector to throw the pictures, if not too large, onto a screen.

Advantages and Disadvantages. The opaque projector has certain advantages. It will project, by reflected light, any opaque flat or nearly flat material, such as drawings, graphs, or diagrams, and any written or typed material, such as lists of examples and problems. It will reproduce color on the screen. It is extremely simple to operate.

It is possible to buy a combination projector for opaque materials and 3¼″ x 4″ standard glass slides. The change from one to the other type of projection can be made instantly. By means of suitable accessories, this combination projector can also be adapted for the projection of 2″ x 2″ glass slides and 35 mm. filmstrips.

Because the opaque projector, however, projects by reflected rather than transmitted light, much of the light is lost. It is, therefore, necessary to have the room thoroughly darkened.

Another disadvantage is that the opaque projector is rather bulky in shape. This is not a serious consideration so far as portability is concerned because the newer models are lighter in weight than the older ones.

THE GLASS SLIDE

Description. The glass slide is viewed or projected by transmitted light.

The outside dimensions of the standard American glass are 3¼″ x 4″. The sub-standard 2″ x 2″ glass slide is also being used because it is somewhat less expensive to produce and requires less space for storage. The smaller slide, however, offers less surface on which to work and must be made more carefully. The 2″ x 2″ glass slide must be magnified more diameters than the larger one in order to obtain a screen image of the same size; therefore defects in the smaller slide will become more apparent. Other things being equal, it will also be necessary to use a more powerful lamp in the projector with a 2″ x 2″ slide to obtain a screen image of the same size and brilliancy as that obtained with the 3¼″ x 4″ slide.

The glass slide usually consists of the following items: (a) the glass plate or other material containing the subject to be projected; (b) a mat to mask off a picture area no larger than the aperture gate of the projector; (c) cover glasses to protect the picture surface; (d) binding tape to bind together the several parts of the slide; and (e) a thumbmark to indicate the proper position of the slide when placed in the projector.

Sources. The slides may be made by the individual who wishes to use them, or may be obtained through loan or sale from numerous distributing agencies.

The Photographic Glass Slide. Photographic glass slides are made from negatives by contact printing or enlarging. The photographic slide is merely a positive image on a piece of glass coated with a light-sensitive emulsion, just as a print is a positive image on a piece of paper coated with a light-sensitive emulsion. The necessary materials for making photographic slides, with instructions, may be purchased from any of the several companies producing photographic materials. It is possible to tint these photographic slides by means of water colors.

Glass slides are also made from positives on film of the negatives. The positive is mounted between two pieces of cover glass for projections. The technique is the same in both cases; it is only the type of material used for making the positive that is different.

The Handmade Glass Slide. By a handmade glass slide is meant a non-photographic slide. Writing or drawings for projection may be placed on such media as clear glass coated with clear shellac, etched glass, or lumarith. Lead pencil, colored pencil, India ink, or colored ink may be used. These slides may be mounted in the usual manner, or if they are for temporary use only, they may be mounted temporarily or not at all.

Cellophane will take India ink and can also be used for making handmade slides. It is particularly satisfactory in making typewritten slides. Simply place a sheet of cellophane of the proper size into a folded sheet of carbon paper and then type onto the cellophane through the carbon. Removing the ribbon will produce a better impression. The cellophane slides can be mounted permanently or temporarily.

Slides made by pasting cut-outs of opaque paper onto ordinary cover glass offer interesting possibilities. When projected onto the screen, a silhouette effect is obtained.

Method of Use. The glass slide can be viewed by transmitted light for individual use or projected by transmitted light onto a screen for group use. Standard 3¼″ x 4″ slides are projected by means of 3¼″ x 4″ slide projectors and the smaller 2″ x 2″ slides by 2″ x 2″ projectors. It is possible to adapt the 3¼″ x 4″ projectors for projecting 2″ x 2″ slides.

Advantages and Limitations. Glass slides offer maximum brilliance on the screen. They may be left in the projector for any length of time and may be used in a room not completely darkened. They may be conveniently arranged for use in any desired order.

Glass slides are easily made by the individual who wishes to use them.

There are, however, certain limitations. The slides require considerable storage space, and they are easily broken. They are fairly expensive, although not unduly so in relation to their value.

THE 35 MM. FILMSTRIP

Description. The filmstrip consists of a sequence of pictures on a strip of 35 mm. film, either black-and-white or color, made in any miniature camera using 35 mm. film. The pictures may be either of single-frame size, ¾″ x 1″, or double-frame size, 1½″ x 1″.

Pictures made on natural-color film are returned processed as positives on film, ready for viewing or projection. Pictures made on the usual black-and-white film are developed as negatives, from which as many positive copies on film as desired may be made by contact printing. There is now available a black-and-white reversal film which, when processed, results in positive images on the film and thus eliminates the intermediate step of making negatives.

Method of Use. Because these 35 mm. pictures are small, it is customary to project them. Any projector for 2″ x 2″ slides may be used for the purpose if it is equipped with a 35 mm. filmstrip attachment. The standard 3¼″ x 4″ glass slide projector can also be used by fitting it with one of these attachments.

Sound Filmstrips. Sound filmstrips can be prepared by taking a series of still pictures on 35 mm. film and recording the necessary explanatory material on records. If recorded and run at 78 r.p.m., the record may be used on any standard phonograph as the pictures are projected.

Advantages and Disadvantages. These filmstrips are easily made. They are economical to make or buy or ship from place to place.

They are light in weight, and require very little storage space. A roll of eighteen to seventy-five pictures may be kept in a small can.

There is no danger of getting the pictures out of order, and there is no danger of breakage.

On the other hand, the film is easily damaged because it is unprotected. Furthermore, it is impossible to vary the order of presentation of the individual pictures except by skipping back and forth. These two limitations can be overcome, however, by cutting apart the several frames and mounting the individual transparencies as 2″ x 2″ glass slides.

THREE-DIMENSIONAL PICTURES

Description. The three-dimensional picture, known as a stereograph, has the desirable feature of depth, thus adding reality to the visualization. This advantage is of importance in any situation, but particularly in solid geometry, where many students find it hard to visualize three dimensions from two-dimensional drawings.

Method of Production. There are special two-lens cameras for producing three-dimensional pictures. However, these pictures can also be made on regular black-and-white or color film, in the same way and just as easily as two-dimensional pictures, with any still camera fitted with a stereo attachment over the lens to give the necessary two images on the same piece of film. The film is processed in the usual manner. The 35 mm. miniature still camera is particularly well adapted for producing stereographs inexpensively in this way.

Method of Use. For individual use, the resulting stereographs may be viewed as positives on films (either in filmstrip form or mounted as slides) or as positives on glass or paper (either contact size or enlarged) through suitable stereo viewers.

For group use, 35 mm. stereographs, either in filmstrip form or mounted as 2″ x 2″ slides, may be projected onto a screen by means of projectors made or adapted for the purpose and viewed through suitable spectacles.

MOTION PICTURES

The motion picture, silent and sound, is so familiar that little need be said about it, except that it should not be used for instructional purposes unless motion is essential. Otherwise, the still picture will serve better and is cheaper.

The question of silent versus sound pictures is still a controversial one. Economy and ease of production would seem to indicate that the silent film, either with titles or with spoken comment by the teacher, should be used in preference to sound film where the dialogue on the sound track offers no distinct advantages. It should be kept in mind that the comments on the sound track cannot be adapted to particular teaching situations, should such adaptation be advisable.

TECHNIQUE OF TEACHING WITH VISUAL AIDS

The technique of teaching with visual aids divides itself naturally into three stages: the teacher's preparation, the presentation, and the follow-up.

The Teacher's Preparation. The teacher must consider these steps in preparing to use visual aids: (a) Familiarize himself with the content of the visual material. (b) Decide upon the purpose or purposes in using the aid—introducing the unit, direct teaching, summarizing or reviewing, enrichment, or appreciation. (c) Decide at what point or points in the unit to use the aid. (d) Determine what the students should look for during the showing of the material.

The Presentation. The presentation may be motivated by one or more devices such as these: (a) by developing the need for it; (b) by taking advantage of the interest or previous experiences of the students; (c) by advance assignments—individual, group, or class—based upon what is to be seen.

The students must understand clearly exactly what they are to look for during the presentation. Whether the points on which they are to concentrate their attention be general or detailed, these points should be specifically stated in sentence form on the blackboard or in notebooks. It is advisable to drill upon difficult and new words that may be found in the material to be presented. It is also wise not to list too many points for observation during any one showing.

If at all possible, the material should be presented immediately after developing the points to be observed during the showing.

(a) All apparatus should be ready before the presentation, and arrangements made for operating it. (b) If the material is to be shown more than once, it is wise not to interrupt by calling attention to special items during the first showing. (c) It is advisable, if it can be arranged, to preserve normal conditions by having the class meet as usual in the regular classroom for the presentation.

The follow-up. The discussion of the film should be started as soon as possible after the presentation is completed, preferably the same day, and should be based upon the points on which the students were to concentrate their attention. At the same time it will be possible to answer other questions arising from the discussion and to eliminate any erroneous impressions created during the presentation.

The teacher should capitalize upon interests aroused by the visual material and the discussion to stimulate the students to further activity associated with the unit under consideration.

If the material is presented for the purpose of imparting information, the students should be held responsible for the information gained.

All the material, or only portions of it, may be shown several times, depending upon the needs of the group. Each such subsequent showing should have for its purpose the settling of special problems or the clarifying of conceptions. In these cases it is not necessary to show all of the material.

The technique here presented is simple and easy to follow, but it may mean the difference between success and failure. It really works.

Appendix

APPENDIX I

SHORT DESCRIPTION OF INDIVIDUAL MODELS AND DEVICES

ARITHMETIC

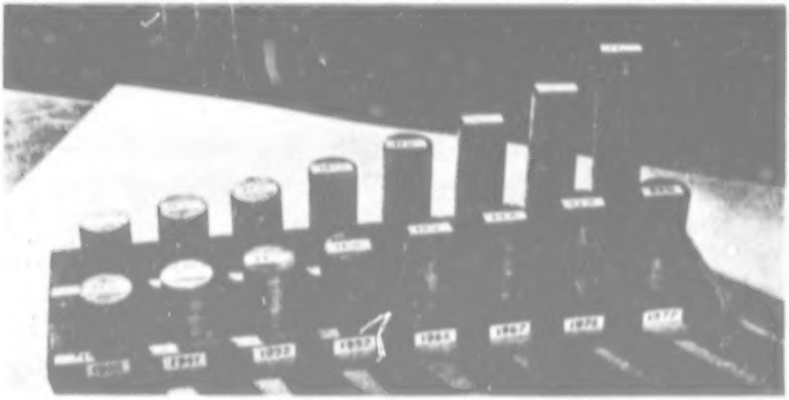
Addition combinations. Slots are cut into paper pasted on cardboard. Numbers appear above slots. Colored slips are inserted as soon as the pupil has had the combination. If he fails one in drill, the colored slip is reversed and the notched end inserted.

ETHEL WEYMOUTH

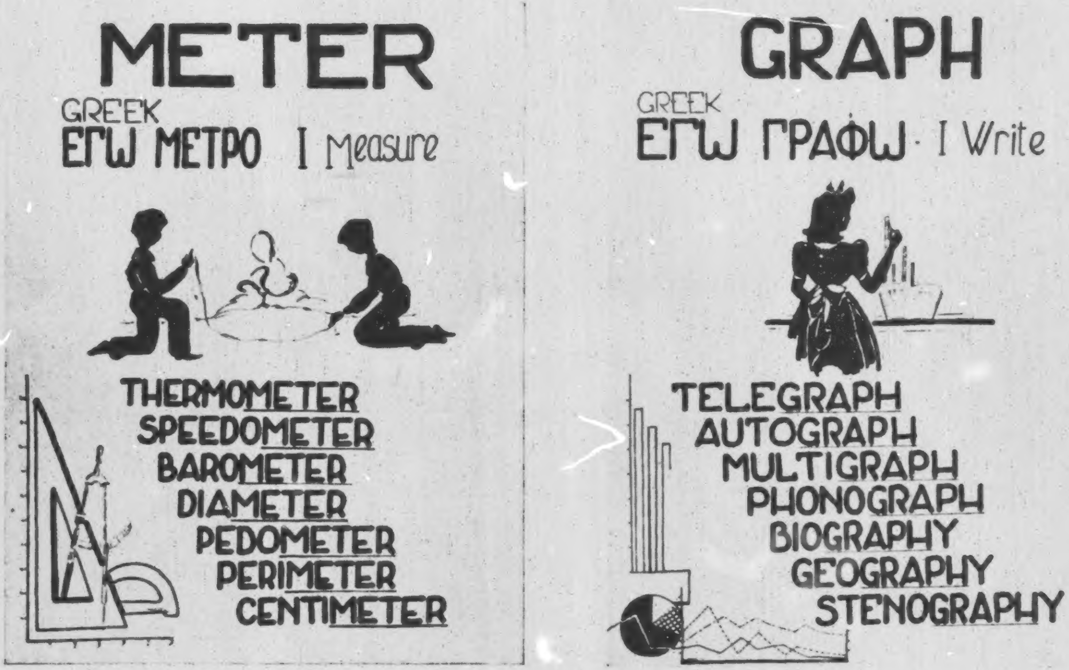
Individual errors in basic combinations. When a child makes an error in addition, subtraction, multiplication, or division, a label with the combination written on it is pasted on a colored piece of paper about the size of a regulation notebook. The child keeps the card in her notebook until someone else misses a combination. A child must give all the combinations on the card before handing it to another child.

ALICE M. HACH

Simple and compound interest. This model, made in the manual training department, shows the difference in the value of money invested at simple interest as compared to compound interest. The first column at the extreme left represents $10 deposited in a bank. The front row shows the increase at simple interest; the back row, compound interest. From left to right each column shows the amount at five-year periods. As time passes, the difference which was not great at first becomes increasingly significant. (The columns of "money" can be made on a shop lathe.)



KATHARINE YOUNG


METER
GREEK
ΕΓΩ ΜΕΤΡΟ I Measure
THERMOMETER
SPEEDOMETER
BAROMETER
DIAMETER
PEDOMETER
PERIMETER
CENTIMETER
GRAPH
GREEK
ΕΓΩ ΓΡΑΦΩ · I Write
TELEGRAPH
AUTOGRAPH
MULTIGRAPH
PHONOGRAPH
BIOGRAPHY
GEOGRAPHY
STENOGRAPHY

Posters Designed to Show Root Measuring of Words
(see page 371)

Vocabulary. Since understanding in mathematics depends to a great extent on understanding vocabulary, we have attempted to focus attention on word meaning in several ways. We have found the following to be the most effective ones:

(a) By using pictures to show the meaning of words. These are selected from various sources and can be arranged as a bulletin board exhibit or on posters for classroom display.

(b) By using posters designed to show the root meaning of the word and its wide use as part of other words in the English language. The accompanying illustrations show two such posters.

(c) By printing the vocabulary of each chapter or subject on a roll of paper attached to a window shade roller. This is hung on the wall of the classroom and can be rolled up out of sight when not in use. Such a list is useful for quick reviews.

KATHARINE YOUNG

GENERAL MATHEMATICS

Picture frames. Frames for pictures of mathematicians are made in the shape of triangles or other geometric figures.

BONNIE E. CONE

Tile designs. A bagful of assorted tiles is kept in the classroom. Tile designs are made by plane geometry pupils and fastened to the cardboard with ordinary glue.

WILLIAM P. SMITH

The circle in airplane markings. The wing markings of the airplanes of various nations are painted on three-inch squares of balsa wood. The circle appears to be basic to most of these designs.

WILLIAM P. SMITH

Measure: visualizing rods, yards, feet, and inches. Pieces of heavy clothesline, cut in lengths of the inch, foot, yard, and rod, are hung on the wall at the front of the classroom. Each piece is brightly painted and labeled.

ALICE M. HACH

Measure: perimeter of geometric figures. Each child is supplied with a packet of toothpicks fastened together with an elastic band. These sticks are used to illustrate the perimeter of various plane figures. Such figures as the rhombus, square, pentagon, hexagon, and others may be readily formed.

ALICE M. HACH

Area. A square foot, cut from cardboard and marked off into square inches, is placed in the front of the room for reference when discussing area problems. Each child is given an envelope containing 100 square inches cut from paper. These are placed within the rectangle, as a means of checking the calculations performed to determine the area.

ALICE M. HACH

Formula for the area of the triangle. (See page 372.)

(a) "Area equals ½ the product of the base and the altitude." (Fig. 1) Cut out of paper two congruent triangles and place together in such a way as to form a parallel-

Area of a Triangle

Figure 1

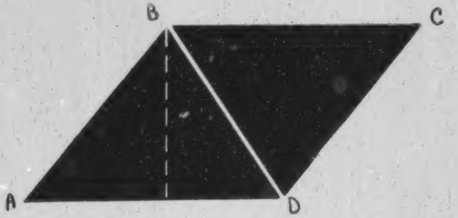


Figure 2

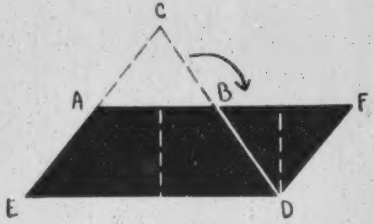
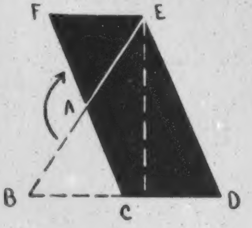
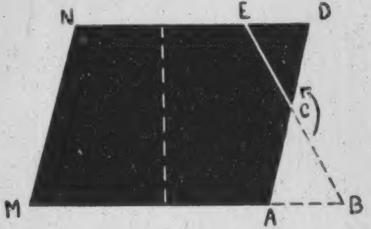
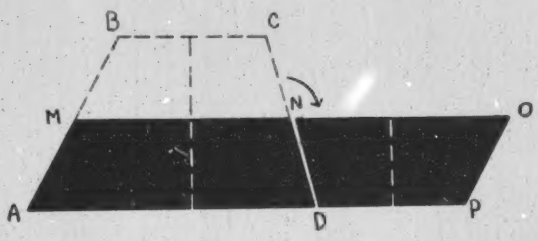


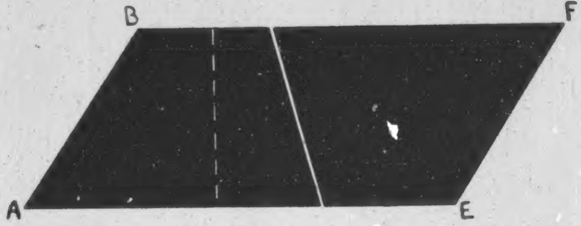
Figure 3



Area of a Trapezoid







ogram *ABCD*. The area of this parallelogram is the product of the base and the altitude. The triangle is ½ the parallelogram, and can be expressed as $\frac{HB}{2}$ or ½ *AB*.

(b) "Area equals the product of the base and ½ the altitude." (Fig. 2) Cut out of paper a triangle *CED*. Cut along line *AB*, which bisects the altitude and runs parallel to the base *ED*. Place *ABC* in position *BFD*, making a parallelogram *AEDF* with base equal to the base of the triangle and the altitude equal to ½ the altitude of the triangle.

(c) "Area equals the product of the altitude and ½ the base." (Fig. 3) Cut out of paper a triangle *EBD*. Cut along line *AC* which bisects the base and runs parallel to *ED*. Place *CBA* in position *FEA*, making parallelogram *CFED* with altitude equal to the altitude of the triangle and base equal to ½ the base of the triangle.

KATHARINE YOUNG

Formula for the area of the trapezoid. (See page 372.)

(a) "Area equals the product of the altitude and ½ the sum of the two bases." (Fig. 1) Cut trapezoid *MNEB* out of paper. Cut along the line *AC* which runs through the center point of *EB* and parallel to *MN*. Place triangle *ACB* in position *DCE*. The parallelogram *NMAD* is formed, whose area is the product of *MA* and the altitude of the trapezoid. But *MA* is ½ the sum of *MB* and *NE*.

(b) "Area equals the product of the sum of the two bases and ½ of the altitude." (Fig. 2) Cut trapezoid *ABCD* out of paper. Fold so that the two bases *AD* and *BC* coincide. Cut along line *MN*. Place upper part adjacent to lower part forming the parallelogram *MAPO*. The area of this parallelogram equals the sum of the two bases multiplied by ½ the altitude.

(c) "Area equals ½ the product of the altitude and the sum of the two bases." (Fig. 3) Cut two congruent trapezoids out of paper. Place them adjacent in such a way as to form a parallelogram *ABFE*. The area of this parallelogram equals the product of the altitude and the sum of the two bases, which makes the trapezoid ½ this parallelogram.

KATHARINE YOUNG

Meaning of the formula for the area of a circle. A wooden box 2" deep is covered with glass. A raised partition divides the bottom of the box through the center. In one side is a depressed circle (Exhibit A, page 374); in the other a depressed area composed of 3 1/7 squares with length of the radius as the side of each square (see Exhibit B). The surface of the circle (see Exhibit B) is completely covered with shot which can be poured over to the other side and shaken down so that it exactly covers the surface of 3 1/7 squares (see Exhibit A). This shows that the actual surface covered by the circle is equivalent to that covered by the squares.

KATHARINE YOUNG

Volume. A cube whose edges are foot rulers or cut from yardsticks is kept before the class for reference in solving problems and as a means of visualizing a cubic foot.

ALICE M. HACH

Geometry project. Each year we have a parents' night at which time we exhibit work of the pupils. One year, instead of the usual projects for the geometry exhibit I asked pupils to bring from home articles in which they could see any geometry. The experiment proved to be successful beyond my expectations. My walls were

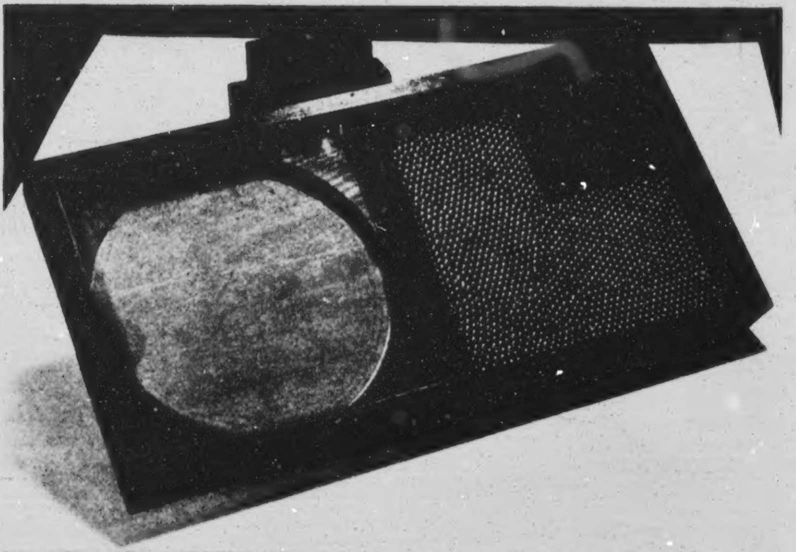


Exhibit A

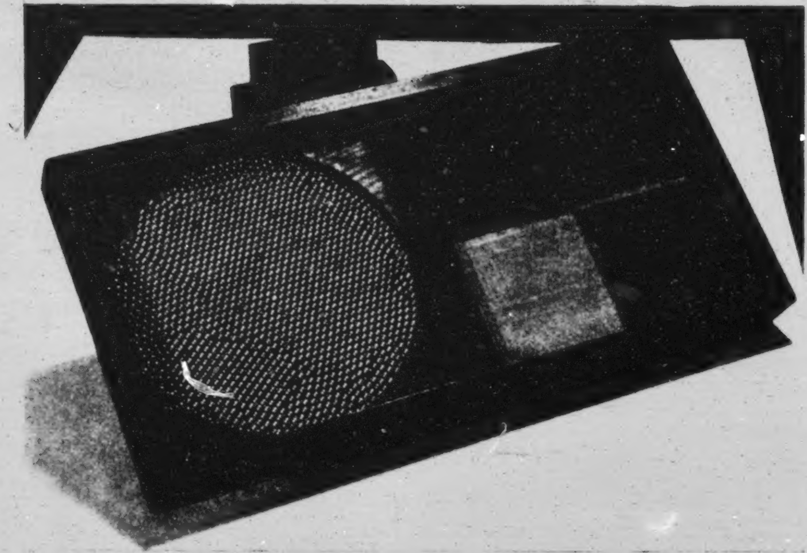


Exhibit B

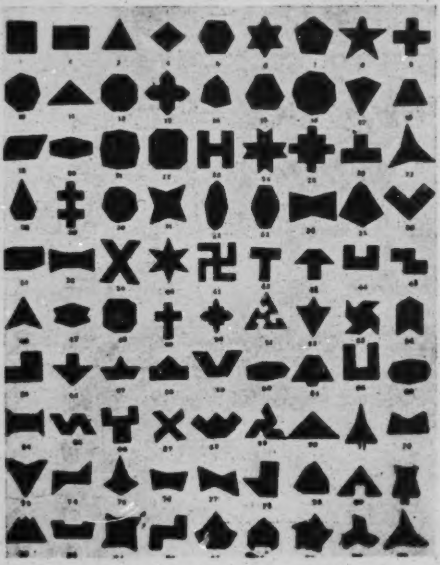
lined with dresses, neckties, shirts, sweaters, and all kinds of wearing apparel; under lock and key in my cupboard were cut glass dishes, silverware, china, jewelry, and many other valuable articles. Some pupils preferred to make scrapbooks. From these they discovered geometry in advertising, in architecture, in the home, in house plans and household appliances, in the arrangement of foods, in linoleum, silverware, jewelry, bedspreads, clothing, in flags and mapmaking, in tractors, transportation, and in nature.

MAE HOWELL

Use of lines and angles in everyday life. After a study of "Geometry in Nature," my classes enjoy several days with the topic "How Lines Are Used in Everyday Life." In a recent class these suggestions were made: (a) Use of line in car design to give appearance of length. (b) Use of concentric circles on hub caps to give appearance of motion. (c) Use of stripes in clothing design to give illusion of height. (d) Use of thick glass or false bottoms in bottles of toilet preparations to give the appearance of large volume. (e) Use of lines and angles to provide optical illusions.

RALPH A. AUSTERMILLER

Polygonal forms. The ninety convex and concave polygons are made of black plastic. The figures are fastened in a plywood frame about 3′ x 4′ and hangs on the wall of the classroom. They are referred to frequently throughout the course in plane geometry. The order of the figures is that assigned by the formula for aesthetic measure of Professor George D. Birkhoff (see *Sixth Yearbook*, National Council of Teachers of Mathematics, pp. 190–195).



WILLIAM P. SMITH

A mathematical Christmas tree. An annual project of the solid geometry classes in the Rapid City, South Dakota, high school has been to make better and more unique solids than those created by previous classes. These included three penetrating cubes, two penetrating pyramids, a star at the top of the tree, a cross, a starred dodecahedron and icosahedron, a snub cube, a truncated icosahedron and fourteen crystal solids.

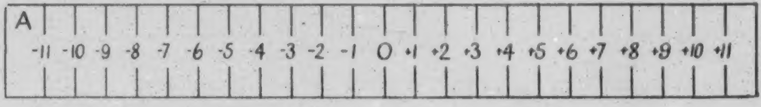
FLORENCE KRIEGER

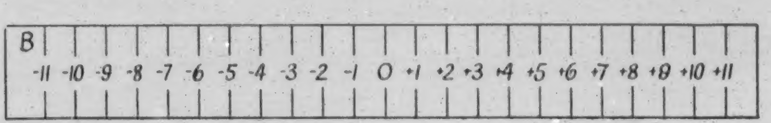
The spiral. The mathematical "sea shell" is etched on black plastic. One side of each right triangle is of unit length. Starting with an isosceles right triangle, whose hypotenuse is $\sqrt{2}$, the successive hypotenuses become $\sqrt{3}$, $\sqrt{4}$, $\sqrt{5}$, $\sqrt{6}$, etc.

WILLIAM P. SMITH

ALGEBRA

Signed numbers—addition. Each pupil makes his own scales of signed numbers and uses them to learn the rules of addition. For example: to add $+5$ and -6, place 0 of scale B below $+5$ of the A scale. The answer will be found immediately above -6 of the B scale, -1.





KATHARINE YOUNG

Model for teaching algebraic multiplication. On page 377 is a device which illustrates the rules for the sign of the product in algebraic multiplication. The apparatus consists of a light wooden bar or lever, balanced at M. Small screw hooks ($r_1, r_2, \ldots, l_1, l_2, \ldots$, etc.) are placed at equal distances to the right and left of M. One end of the string over the pulley has a hook (H) attached, and the other end of the string is left free to be attached to the proper screw hook when a weight is attached at H.

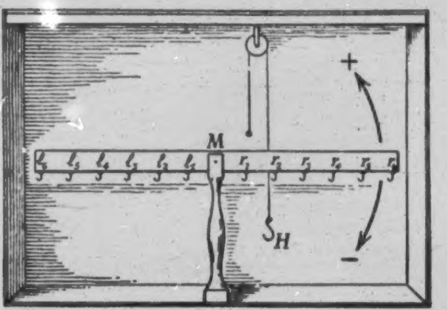
If a weight is hung on a right hook (r_1, r_2, etc.), the bar will turn in the same direction as the hands of a clock, that is, clockwise; a weight hung on a left hook (l_1, l_2, etc.) causes the bar to turn counterclockwise.

Before making the experiments there must be agreement on three matters:

1. Distances on the bar to the right of M are positive; distances to the left of M are negative.

2. Weights attached to the screw hooks (that is, downward-pulling weights) are

negative; weights attached to the pulley string at *H* (that is, upward-pulling weights) are positive.


+
M
l_6 l_5 l_4 l_3 l_2 l_1
r_1 r_2 r_3 r_4 r_5 r_6
H
−

3. Counterclockwise rotation of the bar is considered positive; clockwise rotation of the bar is negative.

The following experiments will show how the apparatus works.

1. To find the product of $+2$ and -4, hang 4 equal weights on r_2. Since the bar turns clockwise, the product is negative; that is, $(+2)(-4) = -8$.

2. To find the product of -2 and -4, hang 4 equal weights on l_2. Since the bar turns counterclockwise $(-2)(-4) = +8$.

3. To find the product of $+2$ and $+4$, fasten the free end of the pulley string at r_2 and hang 4 equal weights on the hook *H*. Since the bar turns counterclockwise, $(+2)(+4) = +8$.

4. To find the product of -2 and $+4$, fasten the free end of the string at l_2 and hang 4 equal weights on the hook *H*, as in Ex. 3. Since the bar turns clockwise, $(-2)(+4) = -8$.

These experiments show that the rules for the sign of the product in multiplication, stated below, are reasonable; that is,

If two factors have like signs, their product is positive; if they have unlike signs, their product is negative.

W. D. REEVE

Graph blackboard. A squared area is drawn on a piece of yellow cellophane, 3¼" x 4", with black and red India ink. If placed between two regular-size lantern glasses and hinged with passe-partout, the cross section lines may be projected on a blackboard with the ordinary slide projector. Curves may be sketched readily on the board. In this way, a graph board may be provided for any classroom.

HENRIETTA TERRY

Graph board. A graph board 5' x 5' is useful when made of wall board which has been painted black and placed in a wooden frame. The squares for the graph are painted with yellow paint. This board is fastened to the wall with wooden pins so that it may be put up when needed and taken down when not in use.

KATHARINE YOUNG

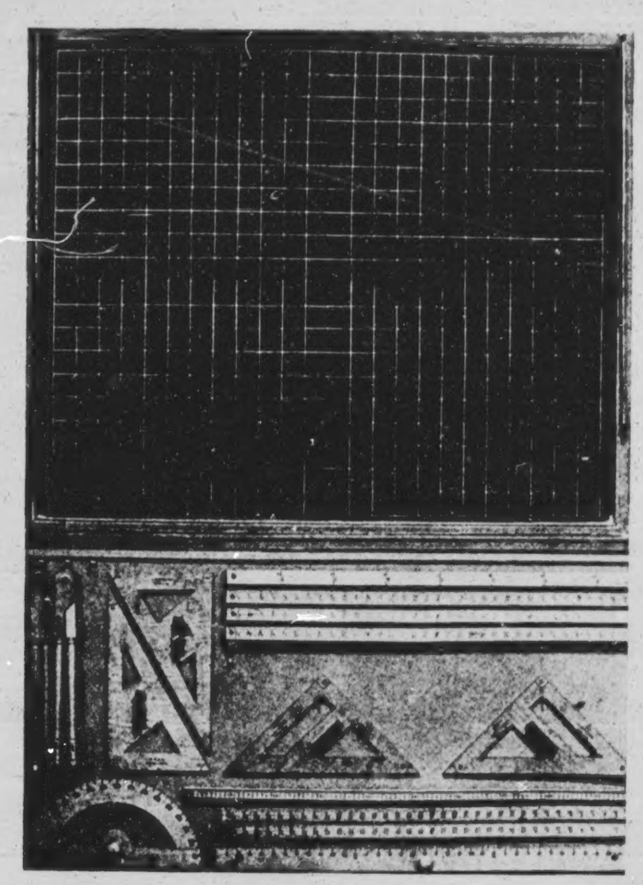
Graph board. One slate of the blackboard is ruled into two-inch squares with the aid of a child's yellow coloring crayon. The cost is relatively little. The lines can be removed at any time, but an eraser does not take them off. If the lines become dim with use, they can easily be retraced.

MARY L. WEBSTER

A graph board—how to produce one in "Barren Budget Land." With the aid of a leather punch, holes are perforated at one-inch intervals on a piece of black oilcloth. The wrong side of the cloth is printed with a crisscross pattern which serves as a guide in keeping the holes equally spaced. A dowel stick added to the top and bottom, or two old curtain rollers, complete the device. When it is placed against the blackboard and a dusty eraser is passed quickly over it, it easily stencils a rectangular system of points.

JAMES R. HAYDEN

Classroom blackboard equipment. The figure here shows a number of instruments and measuring devices that are useful in teaching mathematics.



WALTER BARCZEWSKI

Graphs. Hooke's experiment on elasticity is performed with the aid of a large spring and some weights. A committee of students performs the experiment while the rest of the class records the data. The results are graphed, and the line studied, interpreted, and tested for accuracy of prediction.

Following work on straight line graph, a pan of boiling water is brought into class. At regular intervals a large thermometer is thrust into the cooling water. The temperature drop is recorded and the results are graphed, interpreted, and compared with other records.

LAVERN TRIPP

PLANE GEOMETRY

Use of colored chalk. Relationships in families of theorems may be shown by the use of colored chalk or Ditto ink in marking corresponding parts.

ANICE SEYBOLD

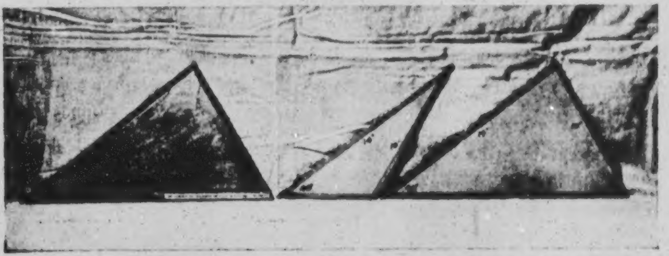
Congruence theorems—using color.* Colored paper parts may be use for the superposition of congruence theorems. For s.a.s. two equal green strips, two equal yellow strips, and two red angles are first exhibited. One triangle is completed by attaching it to the board with transparent tape. The second is completed on top of the first in the order in which the steps of proof are written out.

ANICE SEYBOLD

Congruence of triangles. In discussing congruence and inequality theorems a model is used consisting of a black rigid triangle to which is fastened a light colored flexible triangle. The base of the flexible triangle is equal to that of the rigid triangle and is fastened to it. The left side of the flexible triangle equals that of the black one. The flexible triangle, hinged at two vertices, may be used to illustrate congruence when three sides are equal respectively, as well as when two sides and the included angle are equal. Inequality theorems may also be illustrated.

H. FUHRMANN

Congruent triangles (ambiguous case). Two triangles are not always congruent if two sides and an angle of one are equal to the corresponding parts of the other. The model shown here is constructed of painted plywood and colored cardboard.



FRANCES M. BURNS

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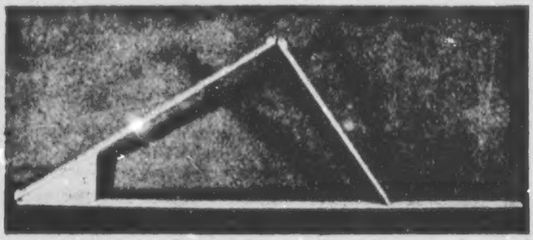
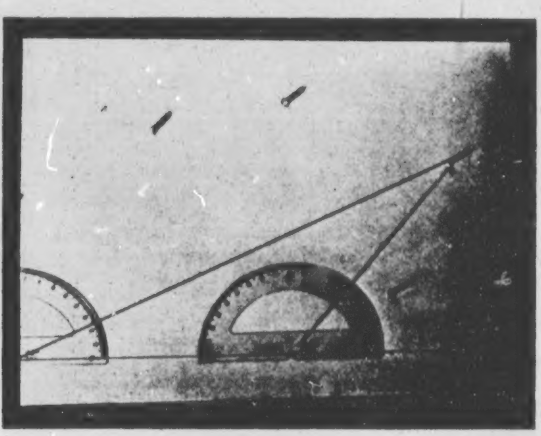
* The trend now is to postulate these theorems. EDITOR.

Triangles. Three strips of wood are fastened together by small metal hinges. The sum of the two shorter strips equals the longest strip. The wooden angle piece is held in place by the weight of the two upper strips. The following may be demonstrated: (a) The ambiguous case in trigonometry. (b) The sum of the two shorter sides of a triangle must be greater than the longest side. (c) The various kinds of triangles.



WILLIAM P. SMITH

Exterior angle of a triangle. An exterior angle of a triangle is greater than either opposite interior angle.



Two 8″ protractors are fastened at each end of the base of a triangle which has been drawn on painted plywood 16″ x 27″. An elastic cord leading from holes at each end of this base can be looped over pegs which locate the vertices of an equilateral triangle, two right triangles, and three obtuse triangles. Readings on the protractors when the vertex is in these various positions show the constant relation between an exterior angle of a triangle and one of its non-adjacent interior angles, and the changing relation between the exterior angle and its adjacent interior angle.

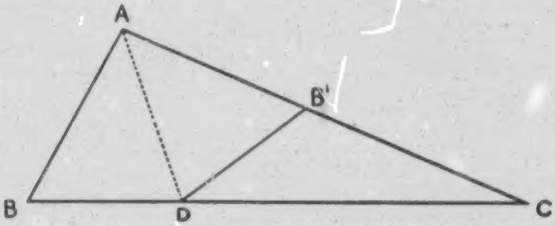
FRANCES M. BURNS

Sum of the angles of a triangle. The sum of the angles of a triangle is equal to 180°.

The two base angles of a plywood triangle having a base 17" and an altitude 11" are cut with a jig saw and fastened with Scotch tape, in their original position, or at the vertex of the triangle where their exterior sides form a straight angle.

FRANCES M. BURNS

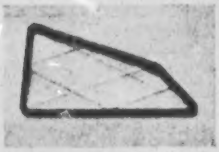
Inequalities. If two sides of a triangle are unequal, the angles opposite are unequal in the same order.

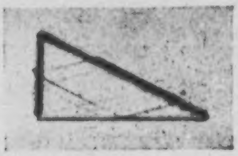


Take a paper triangle *ABC* with unequal sides. Fold one side on another from the common vertex *A*. The crease will be the angle bisector *AD*. The side *AB* is made to fall along *AC*, *B* falling on *B'*, $\angle B$ on $\angle B'$, and since $\angle B'$ is an exterior angle of $\triangle B'CD$, it is greater than angle *C*.

HARRY SITOMER

The general quadrilateral. Four sticks not equal in length are joined to form a quadrilateral. The resulting figure is not rigid. The diagonals are not equal, and do not bisect each other. The angles are not bisected. However, an elastic cord joining the midpoints of the successive sides forms a parallelogram. The sides of the parallelogram equal one-half the lengths of the diagonals. The quadrilateral may be de-formed in two ways to make a triangle in which the elastic cord remains parallel to the base. The model may also be used to illustrate a *skew* quadrilateral.





H. FUHRMANN

Parallelograms. The following theorems are illustrated by the models shown here:

(1) The diagonals of a parallelogram bisect each other.

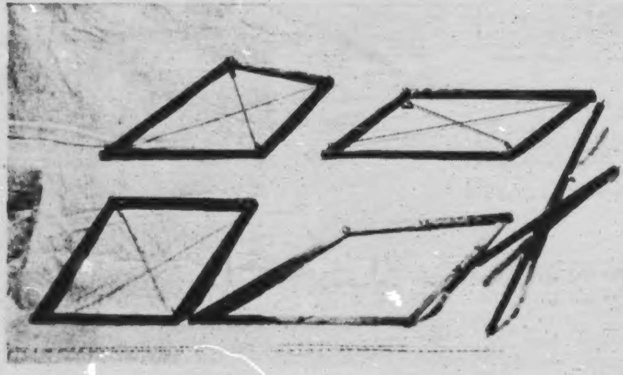
(2) If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.

(3) The diagonals of a rhombus are perpendicular to and bisect each other.

(4) The lines joining the successive mid-points of a quadrilateral form a parallelogram.

(5) Two points each equidistant from the extremities of a line determine the perpendicular bisector of the line.

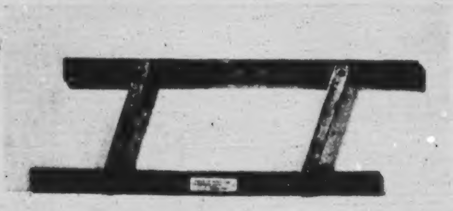
For each of the preceding theorems the hypothesis is executed by means of wooden strips ½" wide joined with round-head stove bolts in such a way that they pivot at the corners. Small elastic cord tied to screw eyes forms the lines for the conclusion.



In addition to the parallelogram theorems, principles such as these can be demonstrated with these models: "Parallelograms with equal perimeters do not always have equal areas," and "The line of centers of two intersecting circles is the perpendicular bisector of their common chord."

FRANCES M. BURNS

Parallel ruler. This parallel ruler for blackboard work is made from the two halves of a yardstick connected by links fastened to the sticks by hollow rivets. It is constructed on the principle, and is a constant reminder of the theorem: If both pairs of opposite sides of a quadrilateral are equal, then it is a parallelogram.



FRANCES M. BURNS

The gyroscopic top. If the top is passed around the class while spinning, pupils feel its reactions to turning. It illustrates the parallelogram of forces.

ANICE SEYBOLD

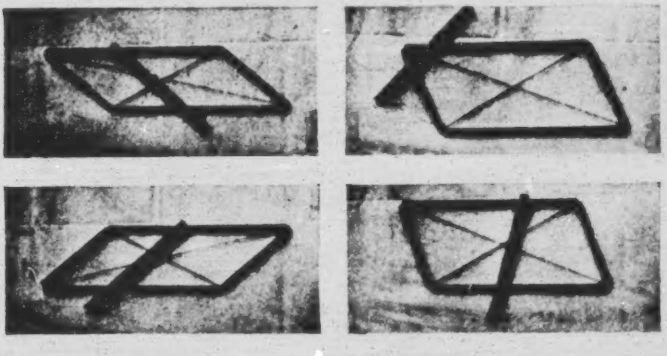
Parallelograms. If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.

Two sticks are crossed and fastened at their mid-point. An elastic cord stretched around the ends assumes the position of a parallelogram. As the angle between the diagonals changes its position, the figure remains a parallelogram. When the diagonals are perpendicular, a rhombus is formed, and the angles of the rhombus are bisected by the diagonals.

H. FUHRMANN

Parallelograms. A flexible parallelogram is constructed from sticks. This illustrates the theorem: If the opposite sides of a quadrilateral are equal, the figure is a parallelogram. Elastic cords joining opposite vertices show that the diagonals bisect each other but that the angles are not bisected. When the diagonals are equal, the parallelogram becomes a rectangle. If another elastic cord is made to pass through successive mid-points of the parallelogram, another parallelogram is formed. This second parallelogram becomes a rhombus when the original parallelogram becomes a rectangle.

H. FUHRMANN



Parallelogram: rigidity. The parallelogram is not a rigid figure. It can change its shape without changing the length of a side. When nails in a crossbar are made to fit into holes in opposite sides of a parallelogram, the bar remains parallel to the other two sides but the figure is not rigid. When the bar is attached to two adjacent sides to form a triangle or a trapezoid with two of the sides, the parallelogram becomes rigid and can support a weight.

H. FUHRMANN

Rectangles. If the diagonals of a quadrilateral are equal and bisect each other, the figure is a rectangle.

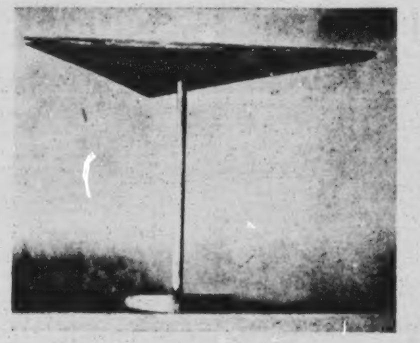
Two equal sticks are fastened at their mid-points. An elastic cord stretched through the ends forms a rectangle. When the sticks are at right angles, the rectangle becomes a square.

H. FUHRMANN

Rectangle. If a rectangle varies so as to assume the shape of a square, the diagonals become perpendicular to each other and resolve into bisectors of the angles of the rectangle.

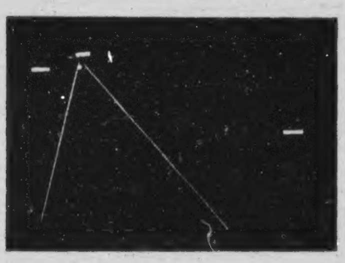
H. FUHRMANN

Median. The medians of a triangle meet at its "center of gravity." The model here shows how the triangle is in balance.



FRANCES M. BURNS

Median. Medians of a variable triangle with fixed base.



The base of a triangle is painted on a piece of plywood 13" x 14". Holes are bored at the extremities of the base and the ends of an elastic cord are knotted at the back. This elastic cord is strung through a screw eye fastened to a peg. A second cord comes from the mid-point of the base; this is threaded and then knotted through a hole in the peg very close to the screw eye. When the peg is fitted into holes representing the vertices of right, acute, isosceles, and obtuse triangles, the position of a median in these triangles is shown.

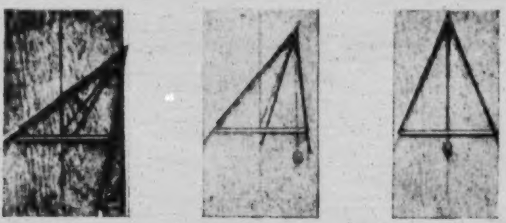
FRANCES M. BURNS

Medians and perpendicular bisectors. Medians and perpendicular bisectors can be shown in a flexible triangle. The perpendicular bisectors of two of the sides may be represented by fixed sticks. The medians to these same two sides consist of elastic cords fastened to the respective vertices and mid-points. The stick representing the perpendicular bisector to the base and the elastic cord joining the mid-point of the base to the vertex of the triangle are shifted to the necessary positions by means of a pulley and carriage arrangement built into the base. This keeps the perpendicular bisector in the center as the base changes in length. As the triangle assumes various sizes, the medians and perpendicular bisectors may be shown to remain concurrent respectively.

H. FUHRMANN

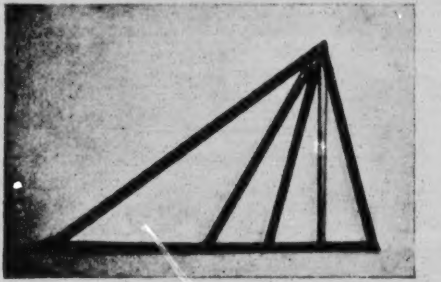
Triangle: altitude, median, perpendicular bisector of side, and angle bisector.

In general, the bisector of an angle of a triangle, the altitude drawn from this same vertex, the median to the opposite side, and the perpendicular bisector of that side are not the same line. The illustrations show a triangle with a constant base, the perpendicular bisector of that base, a movable stick which joins the mid-point of the base to the opposite vertex, an altitude represented by a plumb line, and a bisector of the vertex angle, assuming various or constant positions as the triangle changes from scalene to isosceles. At least six theorems relating to the isosceles triangle may be demonstrated with this model.



H. FUHRMANN

Important lines. Model showing the positions of the altitude, median, and angle bisector of a scalene triangle, base 18", altitude 11". The dowels representing these lines are painted in contrasting colors.



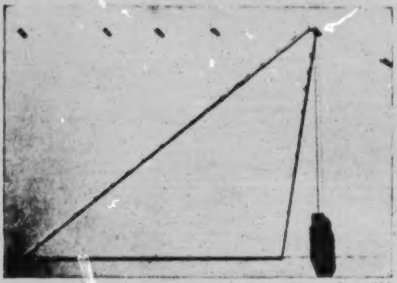
FRANCES M. BURNS

Altitude of a triangle. An altitude of a triangle may fall inside the base, coincide with one side, or fall outside the triangle, depending on whether the angle at the right end of the base is acute, right, or obtuse.

H. FUHRMANN

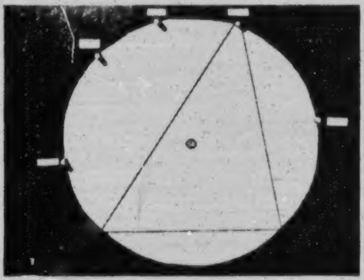
Altitude. Positions of the altitude of a variable triangle having a fixed base.

The base of a triangle is painted on a piece of plywood 12" x 17". Narrow round elastic forming the other two sides leads from holes at the ends of the base and can be looped over pegs located at the vertices of a right triangle, an isosceles triangle, two acute triangles, and two obtuse triangles. When the model is held upright, a plumb bob suspended from the vertex shows the position of the altitude of that triangle with respect to the base.



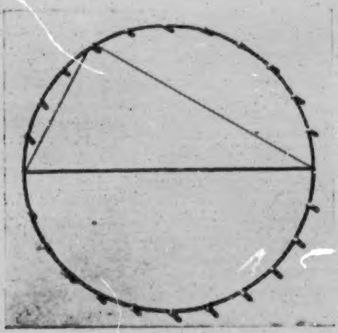
FRANCES M. BURNS

Angles inscribed in a circle. A circle having a 10" diameter is painted on plywood. The center and a chord are marked. A narrow elastic leads from the extremities of the chord to pegs at the vertices of a right triangle, two acute triangles, and two obtuse triangles. When the elastic is stretched over the pegs, the position of the center of the circle in the different kinds of triangles is shown. This model will also demonstrate (a) an angle inscribed in a segment whose arc is less than a semicircle is obtuse; (b) an angle inscribed in a segment whose arc is more than a semicircle is acute; (c) the median to the hypotenuse of a right triangle is one-half the hypotenuse; and (d) angles inscribed in the same segment are equal.



FRANCES M. BURNS

Right angles inscribed in a semicircle. A 12" circle and its diameter are painted on plywood. Narrow round elastic leading from the back through holes at the



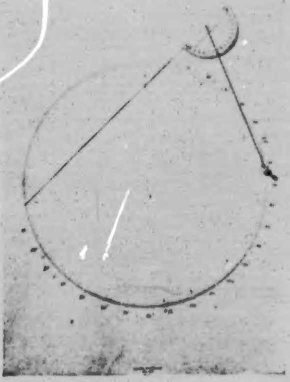
ends of the diameter can be stretched around any of the many pegs fastened on the circle. An angle inscribed in a semicircle is a right angle. The locus of the vertices of the right angles of right triangles having a given line segment as hypotenuse is a circle whose diameter is the given hypotenuse.

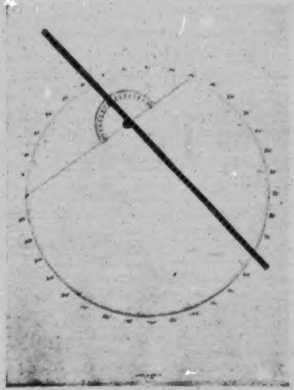
FRANCES M. BURN

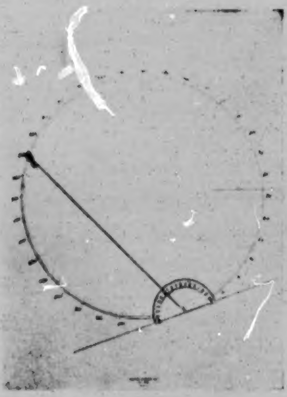
Measurement of angles. Models showing the relation between certain kinds o angles in angle degrees and their arcs in arc degrees are reproduced below and a the top of p. 388.

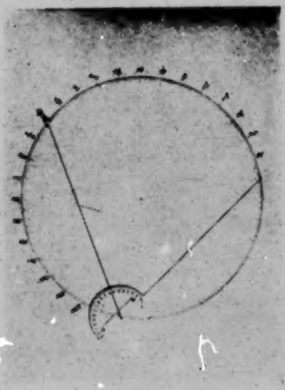
Complete 360° bristol board protractors 14" in diameter are bound to plywoo with Scotch tape. Small ten-cent protractors are attached at the vertex of the angle one side of which is a fixed line drawn on the figure and the other an elastic which can be held at any desired position by a push pin through a small loop at the en of the elastic.

FRANCES M. BURN





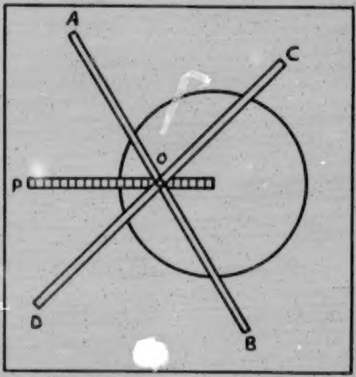




Measurement of angles: circle. A circle with its diameter equal to that of a board protractor is cut from plywood. Its edge is marked at five-degree intervals. Thin steel or copper rods may be placed in such a way as to form central angles, angles formed by two chords of a circle, inscribed angles, and exterior angles. These cases may be shown to merge into one another and hence illustrate continuity. The necessity of adding arcs or subtracting them from each other is also noted.

H. FUHRMANN

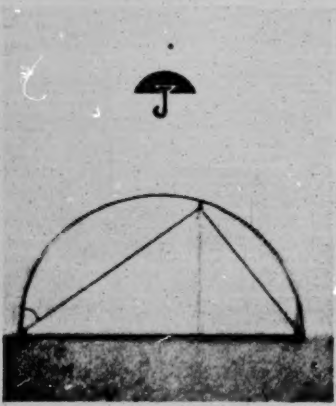
Continuity: angle formed by two straight lines intersecting a circle. Use a piece of cardboard, 20″ x 30″. Cut a slot from point *P* to the center of the circle and



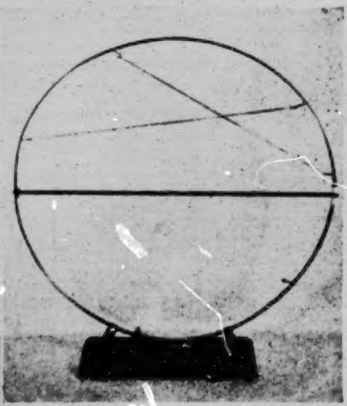
rivet two narrow bars at *O*. These bars slide back and forth in the slot from *P* to the center and illustrate the seven cases in succession.

EMMA HESSE

Theorems relating to chords, arcs, central angles, and inscribed angles of a circle. In model A, a semicircle is made of a chromium strip taken from the side of a wrecked automobile. The full circle in model B is made from a piece of metal molding. The edges of the metal molding are bent in such a way as to form a slot in which slides a nail head as shown in the black cardboard cross section in the picture for model A. Dotted construction lines are shown by means of red rubber bands and the usual sides of the figure with black bands. About fifteen theorems relating to chords, arcs, and central and inscribed angles of a circle can be demonstrated with the aid of these models.



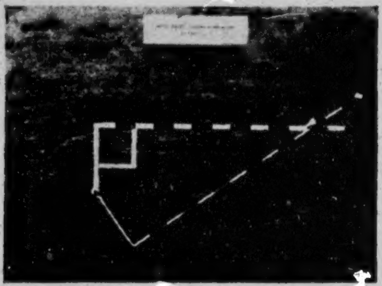
A



B

WILLIAM P. SMITH

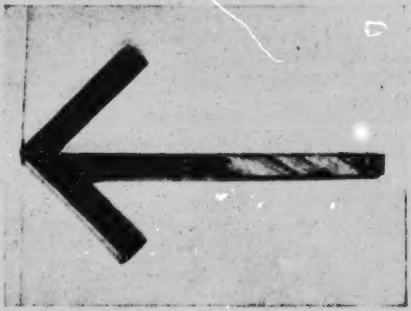
Tangents to a circle. A circle is drawn on a piece of plywood 15" x 20". At the center of the circle is fastened a wire which is bent at a right angle at the outer extremity of the radius to form a tangent. By pivoting the wire through the center



this tangent can have its point of contact at any point on the circle. The movable tangent and a fixed tangent which is painted on the plywood are scaled in bands of contrasting color one-inch wide. The model is used for demonstrating the following relations: (a) Tangents to a circle from the same external point are equal. (b) Tangents at the extremities of a diameter are parallel. (c) If a line is perpendicular to a radius at its outer extremity, it is a tangent.

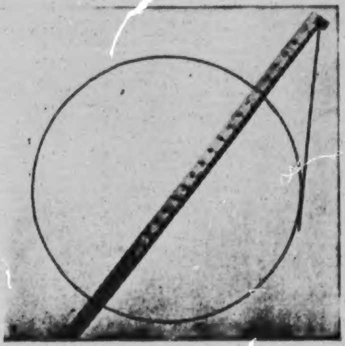
FRANCES M. BURNS

Center-square, 15" long, for finding the center of a circle. This is used to help apply the following theorems: (a) Tangents to a circle from the same external point are equal. (b) The bisector of the vertex angle of an isosceles triangle is the perpendicular bisector of the base. (c) The perpendicular bisector of a chord passes through the center of the circle. The upper edge of the bisector always bisects the angle.



FRANCES M. BURNS

Tangent-secant relation. A large circle is drawn on bristol board backed with plywood. The tangent is fixed, while the secant, marked in ¾" units, pivots from an



external point. If care is taken in choosing the dimensions, several illustrations of the tangent-secant proportion using whole numbers can be obtained.

FRANCES M. BURNS

Secant and external segment. A large circle is drawn on colored showcard board mounted on plywood. A 21" secant marked in inches pivots from an external point. If care is used in choosing the dimensions, several illustrations, involving only integers, of the constant relation between the secant and its external segment can be shown.

FRANCES M. BURNS

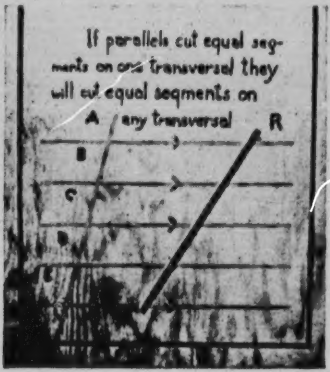
Area of a segment of a circle. Colored cardboard with the chord of the segment scored and taped with Scotch tape so that the triangle will bend out of sight illustrates the principle that the area of the segment equals the area of the sector minus the area of the triangle.

FRANCES M. BURNS

Similar triangles (blackboard device). A triangle having a base 14" and an altitude 10" is cut from plywood. A second triangle whose sides are parallel to and 1¼" from the larger one is removed, giving a tool for quickly drawing similar triangles on the blackboard.

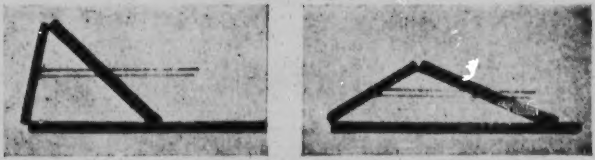
FRANCES M. BURNS

Equal segments. A series of parallel lines which cut equal segments on a transversal is painted on a piece of plywood 16" x 20". The "any other transversal" is represented by a dowel which pivots. This can also be used to illustrate the theorem: If a line bisects one side of a triangle and is parallel to a second side, it bisects the third side.


If parallels cut equal segments on one transversal they will cut equal segments on any transversal
A
R
B
C
D

FRANCES M. BURNS

Triangles. The line segment joining the mid-points of two sides of a triangle is parallel to the third side and equal to one-half of it.



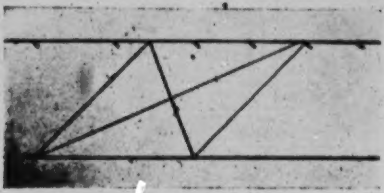
A flexible wooden triangle is used. The base is marked off in equal divisions. A stick attached to the mid-point of the second side has the same divisions. When the triangle changes positions the stick is rotated so that it passes over the mid-point of the third side. It then remains parallel to the third side and equal to one-half of it. (Strips of graph paper may be fastened to the base and stick if more accurate readings are desired.)

H. FUHRMANN

Similar triangles and proportion. A triangle having two of its sides 16" and 12" is painted on plywood. Half-inch units are marked on both sides. These are numbered at intervals of 4" on the 16" side and at intervals of 3" on the 12" side. Elastic cords attached to push pins can be fastened at any desired points. This model has been used to demonstrate the theorem: If a line divides two sides of a triangle proportionally, it is parallel to the third side. It will also show the converse of this theorem.

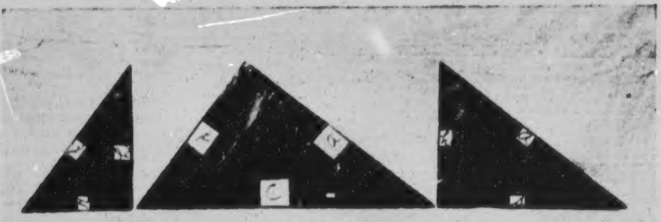
FRANCES M. BURNS

Area of triangles. Two parallel lines 7½" apart are painted on a piece of plywood 12" x 24". A triangle with its vertex in one of the parallels and its base in the other is drawn. Elastic cord leading from holes at each end of the base can be hooked over any of the six pegs in the parallel through the vertex. The positions of the pegs show right triangles, acute triangles, and obtuse triangles, all of which are equal in area to the fixed triangle.



FRANCES M. BURNS

Right triangle proportions. These three plywood triangles help to demonstrate the "right triangle proposition." Dimensions of the largest are: base 20", altitude 9½".



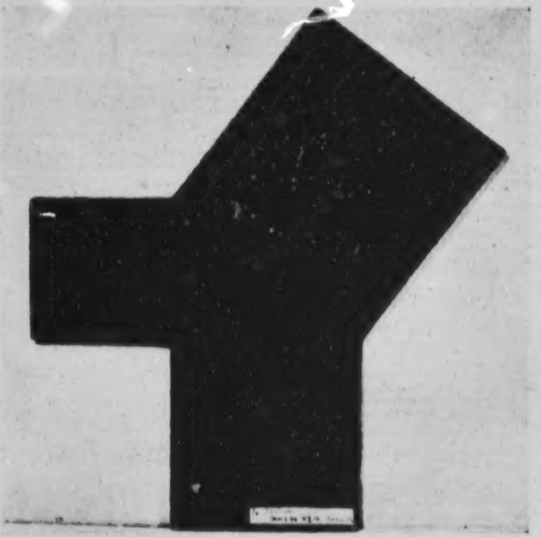
FRANCES M. BURNS

Mid-points of triangles. The line joining the mid-points of the sides of a triangle is parallel to the third side and equal to one-half of it.

A narrow strip of wood to represent the base of a triangle is tacked to painted plywood 11" x 16". A second strip of wood, half the length of the first, is fastened at the mid-points of two unequal pieces of flat elastic cord which form the other two sides of the triangle. A push pin through the vertex places the vertex at any desired point. The line joining the mid-points of the two sides remains parallel to the base, whatever the size or kind of triangle.

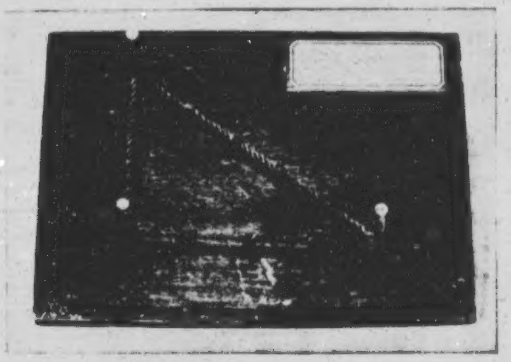
FRANCES M. BURNS

Pythagorean Theorem. The square on the hypotenuse of a right triangle equals the sum of the squares on the other two sides.



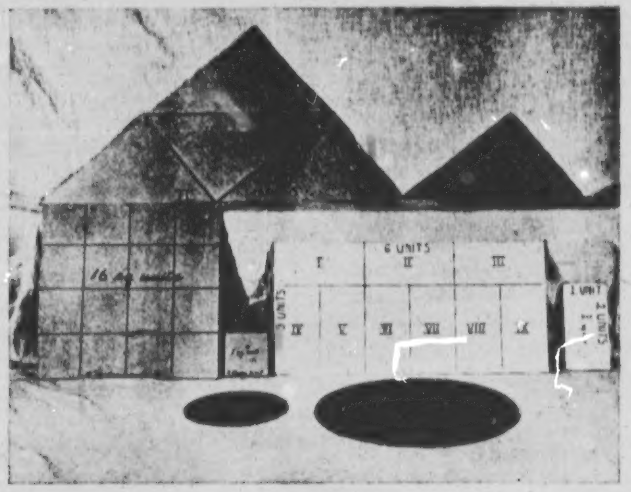
FRANCES M. BURNS

Pythagorean Theorem. An old method of building a right angle by "rope stretching," using the 3-4-5 combination, is illustrated here.



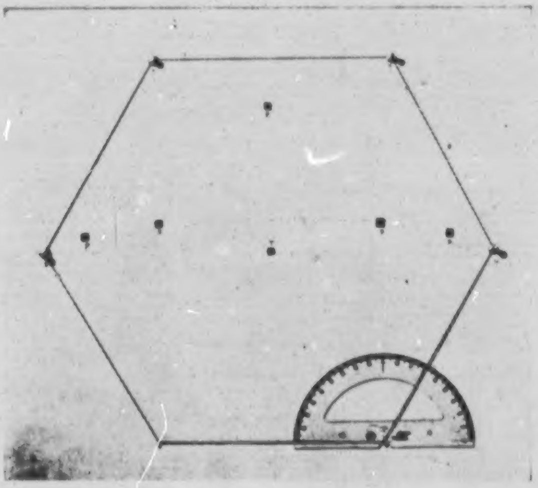
FRANCES M. BURNS

Areas of polygons. The areas of similar polygons and circles are to each other as the squares of their corresponding lines.


16 sq. units
6 UNITS
I
II
III
3 UNITS
IV
V
VI
VII
VIII
IX
1 UNIT
2 UNITS

FRANCES M. BURNS

Interior and exterior angles of polygons. On a piece of plywood 22" x 24" a base line segment 10" long is drawn, having an 8" protractor attached at one end. Vertices of the equilateral triangle, square, regular pentagon, and regular hexagon having the 10" line segment as base are carefully located and marked. Holes are bored at the extremities of this base and small round elastic cord is knotted at the back. When this elastic is held in place by push pins, the various polygons are completed. The model will show (a) the size of each interior and each exterior angle of these regular polygons and (b) how each interior angle increases while each exterior angle decreases as the number of sides of a regular polygon increases.



FRANCES M. BURNS

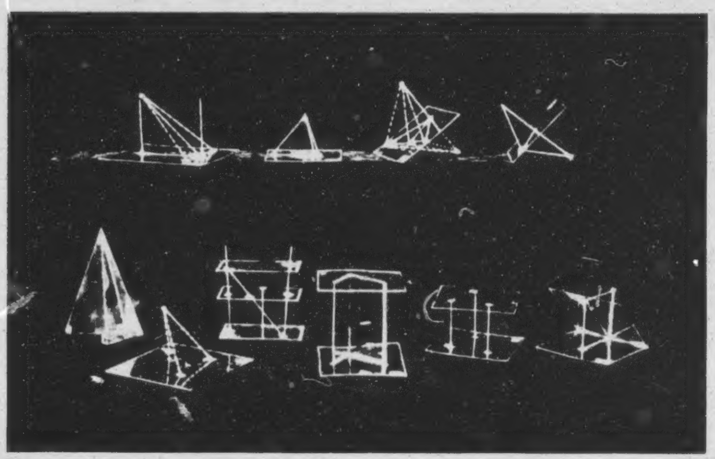
Limit of inscribed polygon. A 14" circle is painted on plywood. Pegs are located at the vertices of the inscribed square, regular octagon, and regular sixteen-sided polygon. When elastic is stretched over the pegs, the theorem: If the number of sides of a regular inscribed polygon is repeatedly doubled, the perimeters and areas approach nearer and nearer to those of the circle, is illustrated.

FRANCES M. BURNS

SOLID GEOMETRY

Solid geometry figures. Seeing three dimensions. A magnifying glass, if held over figures in the textbook for solid geometry theorems, gives an impression of depth in addition to that produced by the ordinary perspective of the drawing or picture.

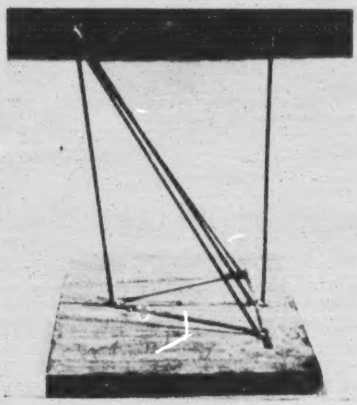
Models for the proofs of propositions in solid geometry. Clear celluloid 1/16" thick is used for the planes, painted applicators for the lines in space, and narrow colored Scotch cellulose tape for the lines on the planes. Duco cement joins the planes; sealing wax holds line to line or line to plane.



Balsa wood models may also be used for proofs of propositions in solid geometry.

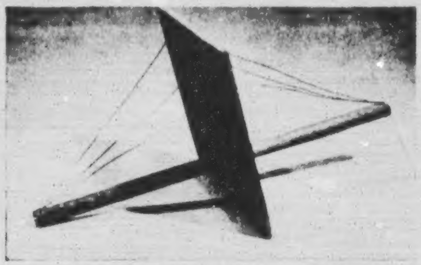
FRANCES M. BURNS

Lines and planes. Lines perpendicular to the same plane are parallel.



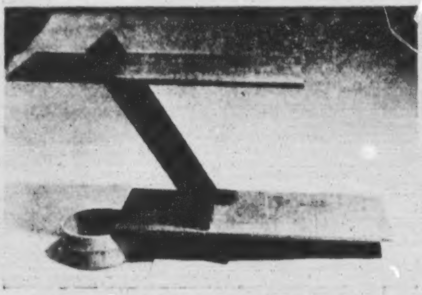
CHRISTINE FISCHER

Lines and planes. If a line is perpendicular to each of two intersecting lines at their point of intersection, it is perpendicular to the plane determined by these lines.



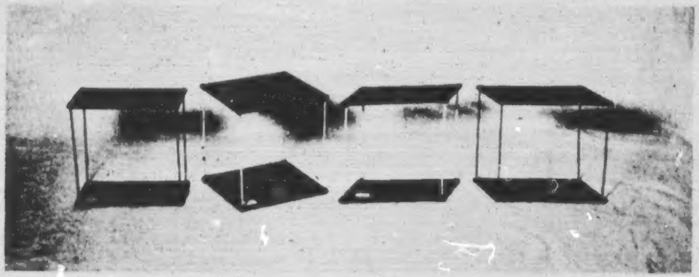
CHRISTINE FISCHER

Planes. If two parallel planes are cut by a third plane, the intersections are parallel.



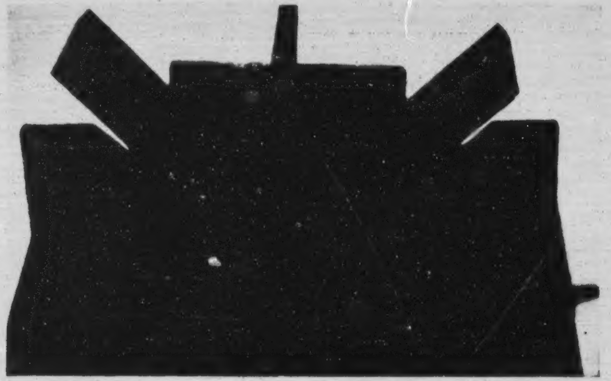
CHRISTINE FISCHER

Parallelepiped. Wire and plywood models may be used to illustrate any right, rectangular, regular parallelepiped.



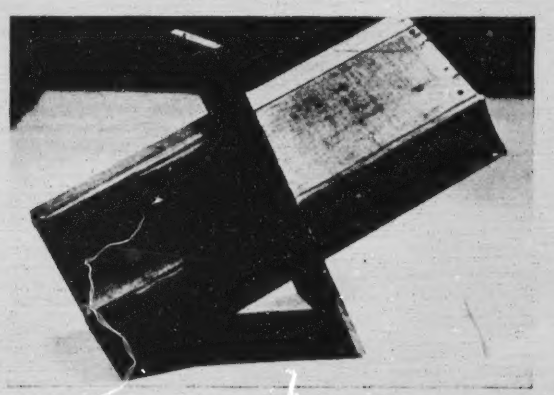
FRANCES M. BURNS

Locus. The locus of points within a dihedral angle and equidistant from the faces is the plane bisecting the dihedral angle.



CHRISTINE FISCHER

Sections. The intersection of a plane and a surface is a section.



CHRISTINE FISCHER

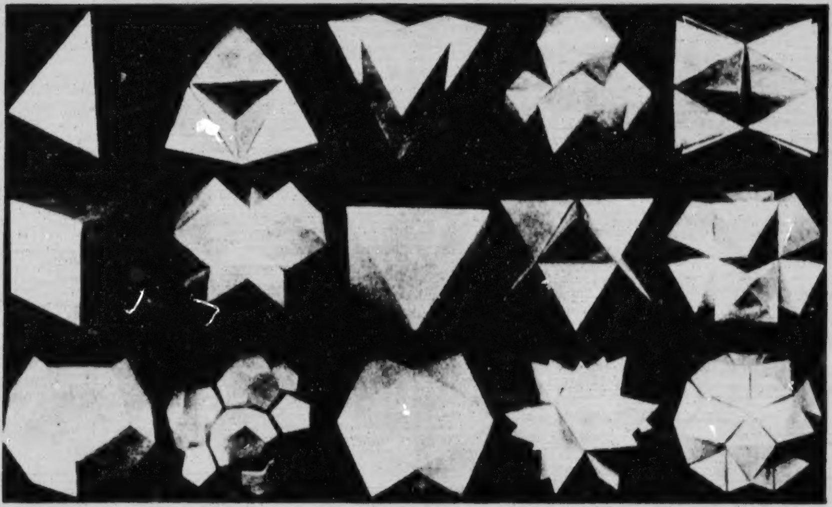
Moravian star. A star polyhedron is formed from an icosahedron. Made from translucent material with electric light within, it is used as a decoration in Moravian homes at Christmas time.

FLORA MAE WATSON

Polyhedrons. Three intersecting cubes, each cube constructed in a different color of cardboard, may be used as a Christmas tree ornament.

FLORENCE KRIEGER

Some composite polyhedrons. The solids shown in the accompanying figures appear in the following order:



Top row: tetrahedron, tetrahedrons on each face of a tetrahedron, octahedrons on each face of a tetrahedron, icosahedrons on each face of a tetrahedron, tetrahedrons on each face of the tetrahedrons on each face of a tetrahedron.

Middle row: cube, cubes on each face of a cube, octahedron, tetrahedrons on each face of an octahedron, octahedrons on each face of an octahedron.

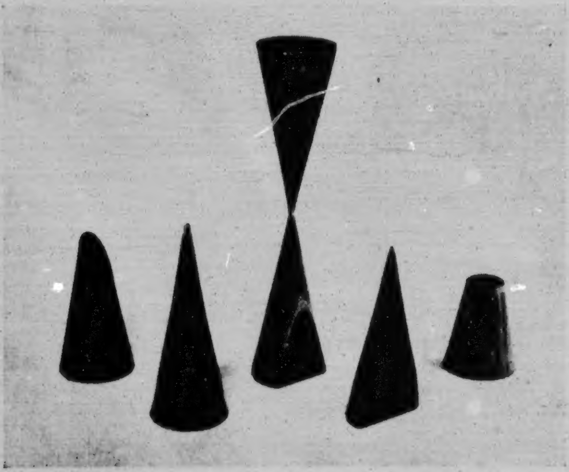
Bottom row: dodecahedron, dodecahedrons on each face of a dodecahedron, icosahedron, tetrahedrons on each face of an icosahedron, octahedrons on each face of an icosahedron.

DWIGHT F. HEATH

Conic sections. A glass cone is sealed at the top. The level of the liquid shows the conic sections as the cone is revolved.

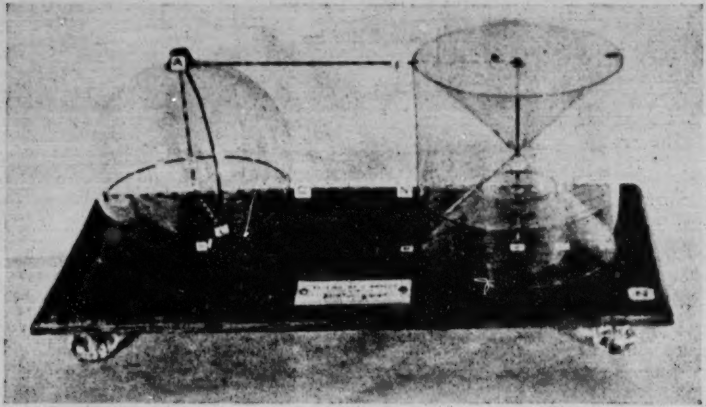
WILLIAM P. SMITH

The sections of a cone. These conic sections were hand turned on a lathe by a student.



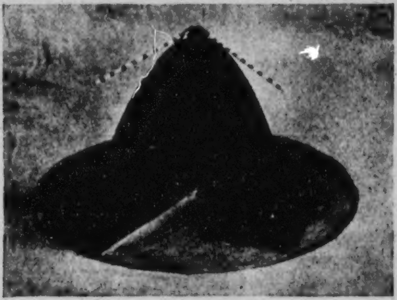
FRANCES M. BURNS

Volume of a sphere, using Cavalieri's principle. The sphere in this model is made of Pyrex glass and the cylinder of celluloid. The diameter of the base of the cylinder and of the cone equals that of the sphere. The altitude of the cylinder equals the diameter of the sphere. The section of a sphere parallel to the basic plane equals the section of the cylinder minus the section of the cone. The boy who made this worked on it for two months during his spare time.



WILLIAM P. SMITH

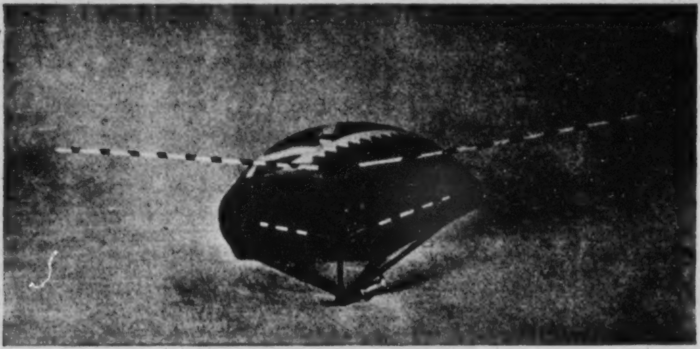
Spherical angles. Model for proof "A spherical angle is equal in degrees to ——."



A quarter circle of 10" diameter is glued to a complete 10" circle. A second quarter circle is hinged to the first in such a way that it turns freely through 180°. The circles are plywood painted in different colors. Applicators at the vertex of the angle are used for tangents.

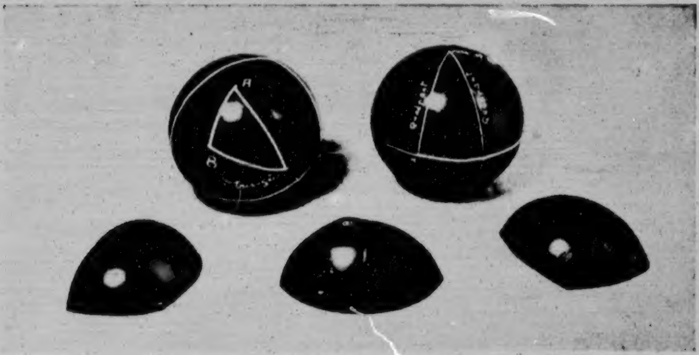
FRANCES M. BURNS

Sides and angles of a spherical triangle. The sides of a spherical polygon have the same measure as the face angles, and the angles of the polygon have the same measures as the dihedral angles of the corresponding polyhedral angle at the center. A spherical quadrilateral is cut from a rubber ball 7" in diameter. Small wires leading from the vertices are soldered together at the center of the sphere. Applicators are used for the tangents at one vertex and for the perpendiculars to the radius.



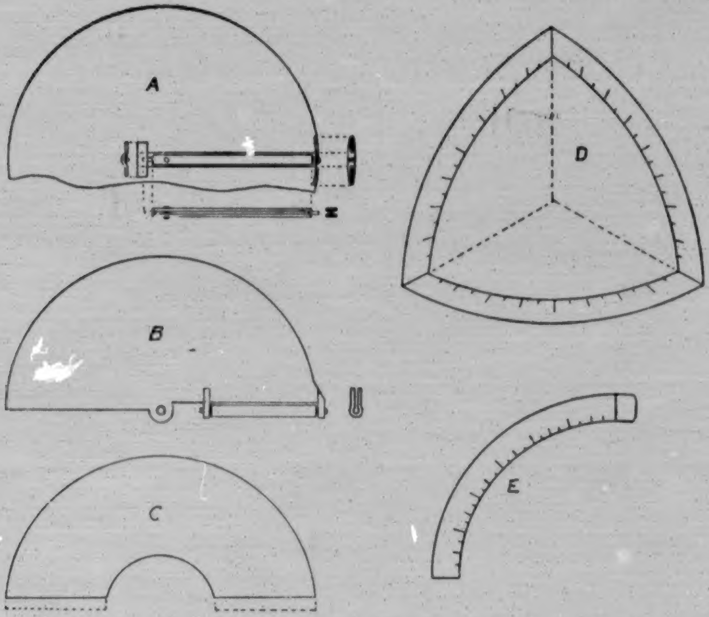
FRANCES M. BURNS

Spherical triangles. Location of the pole of a great circle. Polar triangles. Congruent and symmetric spherical triangles.



FRANCES M. BURNS

Model of spherical triangle. A is the complete base circle, radius 2½″. The slotted bar through which *B* turns is formed of two strips of brass, separated at the ends



by short pins to which they are soldered. The right-hand bearing is the saddle shown in the plan in *A* (p. 402). This bearing is soldered to *A*. The left bearing is a pair of clips, one on each side of *A* and soldered to it. Allow just enough space at each end of the bar for play between bearings. For clarity, the cut shows more. *B* is the second (half) circle, same radius as *A*, turning about a pin in the slotted bar of *A*, at the center of circle *A*. *B* carries a slotted bar, exactly like that of *A*, on a radial line of *B*. Bearings as shown are soldered to *B*. *C* is the third (half) circle sliding in the slotted bar of *B*. To hold the parts in any chosen position a small button rivet may be placed on the inside of the bar slot, or the bar slot may be bent together at the middle to provide friction.

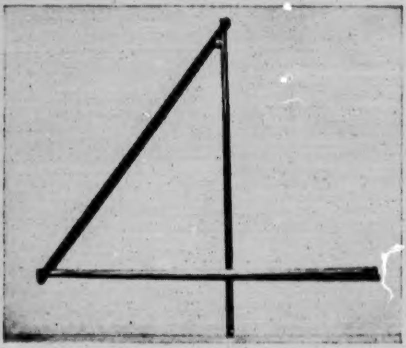
The device may be improved by providing means for measuring arcs and angles. This may be done in any one of three ways: (a) Graduations on edges of circles. (b) Semicircular scale, radius 2½", graduations on inner edge of semicircle. (c) Triquadrantal triangle, with graduations on inner edges of quadrants. *D* is made up of three sections, like *E*, each soldered to the next by the tab, shown on *E*, which has been bent at right angles to *E*.

The circles should be of tin of sufficiently heavy guage to prevent bending.

L. LELAND LOCKE

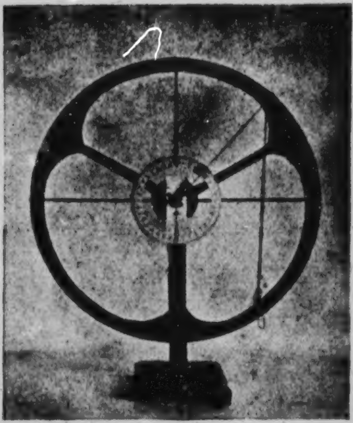
TRIGONOMETRY

Variation of trigonometric ratios. Three 18" colored dowels which pivot illustrate the changes in the trigonometric ratios for angles from 0° to 90°.



FRANCES M. BURNS

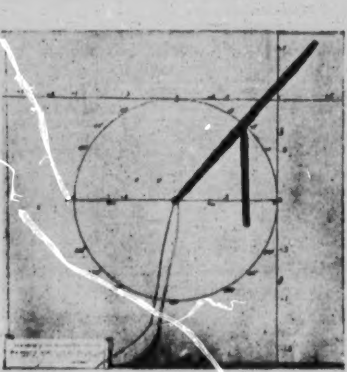
Trigonometric functions of the general angle. A circle is sawed from a piece of mahogany board. A movable steel generating arm has a weighted piece of string fastened to the end of it. As the arm rotates, the functions of angles in the various quadrants are demonstrated. After the arm has passed 180°, the weighted string



must be held up, or a slit must be made along the horizontal diameter of the circle through which the string may be passed.

WILLIAM P. SMITH

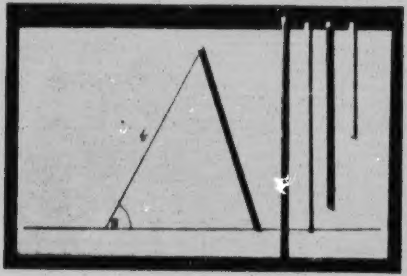
Variation of trigonometric functions. On bristol board, bound to plywood by Scotch tape, a 10″ circle is drawn. At the 0° and 90° points tangents scaled in half units are constructed. A metal strip bound with black adhesive pivots at the center.



To this metal strip, at the outer extremity of the radius, a second movable strip is attached. Each of these is also marked in half units. Special angles are marked on the circle. The board helps to demonstrate the line values of the trigonometric functions, values of these for special angles, and changes in the functions as the angle increases from 0° to 360°. When negative values of the secant and cosecant are needed, an elastic cord is pulled from the back through a hole at the center of the circle.

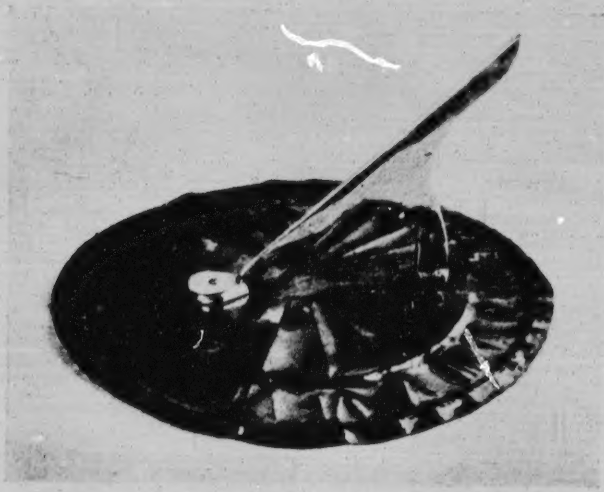
FRANCES M. BURNS

Solution of triangles—ambiguous case. Painted dowels of the desired lengths fit into a hole at the upper end of line segment *b*. Material used is bristol board backed with plywood. Size 16" x 26".



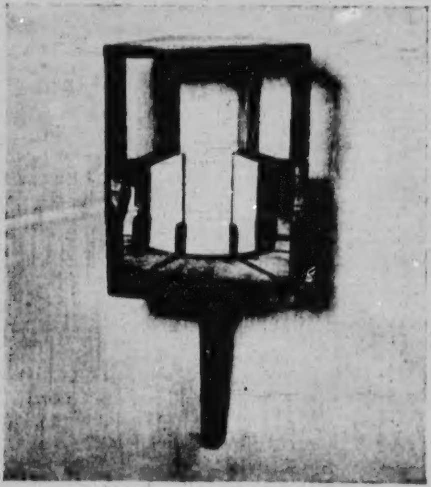
FRANCES M. BURNS

Sun dial. Sun dial for north latitude 40° 6'. Hour lines are located on a copper base. Gnomon is aluminum.



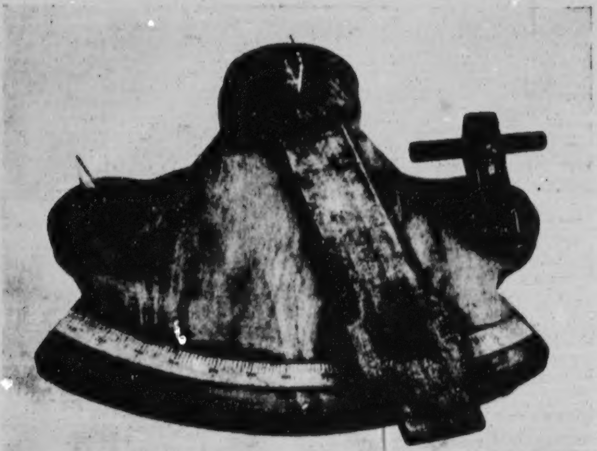
FRANCES M. BURNS

Angle mirror. Plywood and ten-cent-store mirrors. Used to establish a right angle and to lay out a large circle.



FRANCES M. BURNS

Sextant made of plywood.



FRANCES M. BURNS

APPENDIX II

BIBLIOGRAPHY

SPECIAL ARTICLES

ARITHMETIC

A Method of Finger Multiplication. E. J. Rendtorff. School Science and Mathematics, Vol. 8, pp. 580–581, October, 1908.

A method applicable to the products of any numbers, when these, less 1, on being divided by 5 give the same whole number as the quotient.

Resourcefulness in Teaching the New Arithmetic. Harriet E. Glazier. School Science and Mathematics, Vol. 40, pp. 777–779, October, 1940.

Why are our cans made round and so much the same shape? Assignment to make a simple paper model to hold a gallon. Various results are listed, leading to the cylinder whose height equals diameter of the base.

The Rhombic Dodecahedron for the Young. W. Hope-Jones. Mathematical Gazette, Vol. 20, pp. 254–257, October, 1936.

Junior high school students may construct a rhombic dodecahedron as related to bee-cell. Diagrams and directions.

A Device as an Aid in Teaching the Idea of Tens. Herbert F. Spitzer. School Science and Mathematics. Vol. 42, pp. 65–68, January, 1942.

Uses unit blocks and bundles of ten to introduce the child to numbers and their representation.

Construction of a Honeycomb. Louis Vogel. School Science and Mathematics, Vol. 37, pp. 386–387, April, 1937.

Directions and diagram for making cardboard bee-cell to be used at third or fourth grade level.

Methods in Arithmetic and Algebra. R. L. Short. School Science and Mathematics, Vol. 39, pp. 239–250, March, 1939.

Suggestions for helps in arithmetic and algebra from second grade through ninth. Multiplication tables for the tens system and for fractions, pure decimal fractions, easy proportion, understanding easy quadratic methods, easy squares and square roots.

GENERAL MATHEMATICS

Introducing Mathematical Concepts in the Junior High School. David W. Russell. School Science and Mathematics, Vol. 38, pp. 6–19, January, 1938.

How pictures were used to introduce mathematical concepts of perspective, architecture, and applications in engineering, insurance, mining, oil industry, and safety.

Objective Materials in Junior High School Mathematics. Joy Mahachek. Mathematics Teacher, Vol. 32, pp. 274–275, October, 1939.

Devices for measuring angles and distances.

Classroom Practice in the Teaching of Everyday Mathematics I. Raymond J. Mejdak. Mathematics Teacher, Vol. 34, pp. 368–369, December, 1941.

Models are used to find areas of solids, to illustrate theory behind equations, and in the study of addition and subtraction of signed numbers.

Suggestions as to a Course in Mathematics for the Entering Class of a High School. J. C. Packard. School Science and Mathematics, Vol. 6, pp. 292–293, April, 1906.

Outlines activities to be carried on in arithmetic, algebra, and geometry—experiments which aid in further study of the subject.

The Supplementary Project in Mathematics. Charles A. Stone. School Science and Mathematics, Vol. 24, pp. 905–912, December, 1924.

Includes a list of twenty topics which may be used as basis for reports. Also bibliography on history, mathematical recreations, famous mathematical problems, famous mathematicians, books of general interest, surveying, journals.

Correlation of Mathematics with Biography, History, and Literature. Joseph V. Collins. School Science and Mathematics, Vol. 5, pp. 640–645, November, 1905; Vol. 5, pp. 726–730, December, 1905.

References to books dealing with lives of Longfellow, Washington, Lincoln, Napoleon, Grant, Jefferson, Gladstone, Beecher, Franklin, Edison, Wordsworth, Macaulay, and their interest in mathematics. Gives anecdotes which add to class interest.

Mathematics and General Science Cooperate in Junior High School. Jules H. Fraden and Paul M. Tully. School Science and Mathematics, Vol. 40, pp. 541–544, June, 1940.

A sample unit plan on the metric system is shown. Other units suggested are: How fast does sound travel? Measurements of temperature. Science saves health.

Apparatus for Measurement of the Growth of a Plant. George W. Low. School Science and Mathematics, Vol. 5, pp. 27–28, January, 1905.

A simple instrument based on the protractor.

Correlation of the Mathematical Subjects Develops Mathematical Power. Charles A. Stone. Mathematics Teacher, Vol. 16, pp. 302–310, May, 1923.

A graphical solution of a puzzle problem of two candles as solved by a seventh grade student is described and illustrated.

ALGEBRA

An Algebraic Balance. F. C. Donecker. School Science and Mathematics, Vol. 5, pp. 411–415, June, 1905.

Describes a piece of apparatus for demonstrating laws of positive and negative numbers, including laws of signs; commutative, associative, and distributive laws; the transformations of simple equations; laws of the simple and compound levers; law of equilibrium with forces tending to rotate a body; law of parallel forces.

Another Algebraic Balance. N. J. Lennes. School Science and Mathematics, Vol. 5, pp. 602–605, November, 1905.

A demonstration model which can be set up with the aid of stands, metal rod, meter stick, pulleys, and weights.

Geometric Aids for Elementary Algebra. Alberta S. Wannemacher. Mathematics Teacher, Vol. 22, pp. 49–57, January, 1929.

Toothpicks, string, and squared paper are used to study perimeter of regular polygons, draw circles, discover area and volume formulas.

Pictorial Representation of $a^2 - b^2 = (a + b)(a - b)$. Franz Denk. School Science and Mathematics, Vol. 38, p. 686, June, 1938.

Draw square side a. In one corner cut out square with side b. A diagonal cut of remaining piece results in two equal trapezoids which may be rearranged to give rectangle $(a + b)(a - b)$.

Graphical Algebra. Frances E. Nipher. School Science and Mathematics, Vol. 19, pp. 417–420, May, 1919.

Diagrams show how to represent $(y + x)^2$ and $y^2 - x^2$.

Socrates Teaches Mathematics. Norman Anning. School Science and Mathematics, Vol. 23, pp. 581–585, June, 1923.

How wooden blocks are used to represent $(x + y)^2$, $x^2 + y^2$, $x^2 - y^2$, $(x + y + z)^2$.

Graphic Methods in Elementary Algebra. William Betz. School Science and Mathematics, Vol. 6, pp. 683–687, November, 1906.

Lists sequence of topics for introducing graphs and extending their application to algebra and geometry. Bibliography.

Graph Work in Elementary Algebra. F. C. Touton. School Science and Mathematics, Vol. 5, pp. 557–562, October, 1905.

A Graphical Solution of the Quadratic Equation. Albertus Darnell. School Science and Mathematics, Vol. 11, pp. 46–47, January, 1911.

Graphical Solution of Quadratic with Complex Roots. T. M. Blakslee. School Science and Mathematics, Vol. 11, p. 270, March, 1911.

Enrichment Materials for First Year Algebra. David W. Starr. Mathematics Teacher, Vol. 32, pp. 68–77, February, 1939.

Pictures, charts, graphs, motion picture films, periodicals, and list of books for the mathematics library.

High School Algebra. Hiram B. Loomis. School Science and Mathematics, Vol. 7, pp. 590–594, October, 1907.

A Project in Mathematics—Two Algebra Classes Build a Railroad. Donald P. Smith. Mathematics Teacher, Vol. 18, pp. 97–101, February, 1925.

A class of boys and one of girls considered the problems related to building and operating a railway system. A branch of the New Jersey Central was used and information on every phase of its work collected.

Unexplored Possibilities of Instruction in Graphic Methods. William L. Schaaf. School Science and Mathematics, Vol. 41, pp. 160–170, February, 1941.

Suggests that greater use could be made of certain types of curves which rarely appear in secondary mathematics textbooks, of calculating charts or nomographs, and of pictorial graphs and that a critical attitude toward graphs and their interpretation should be cultivated.

Bibliography lists over sixty books and monographs on graphical methods, and periodical references divided as follows: pictorial statistics; graphic methods; application of graphic methods; logarithmic charts; nomography.

PLANE GEOMETRY

Visual Aids in Teaching Geometry. Jessie May Hoag. National Mathematics Magazine, Vol. 14, pp. 153–159, December, 1939.

Some suggestions are: use of colored chalk in helping the pupil visualize correspondence in congruent or similar figures; a small collection of colored cardboard polygons and circles and methods of filing these materials; four rulers bolted together to form a loose-jointed parallelogram; reducing the solar system to scale by drawings, a cardboard model of an old Egyptian level, angle mirror, pantograph, cross-staff, slide rule, sundials. Includes references.

The Use of Enrichment Materials in Secondary Mathematics. Joseph C. Shuttlesworth. Secondary Education, Vol. 6, pp. 210–212, December, 1937.

Students construct regular polygons that fill the plane space around a point and also discover arrangements for regular polygons to fit together to make regular and semi-regular polyhedrons. They also learn to work with surveying instruments and other apparatus. "The usual classroom experience is reversed by proceeding from a concrete and specific problem to the more general geometric theorems which explain such problems."

The Teaching of "Flexible" Geometry. Daniel B. Lloyd. Mathematics Teacher, Vol. 32, pp. 321–323, November, 1939.

Using the pantograph, parallel rulers, transit, and linkages in plane geometry.

Vitalizing Geometry by the Use of Pictures. Donovan A. Johnson. School Science and Mathematics, Vol. 38, pp. 1032–1034, December, 1938.

Topics which can be advantageously illustrated by pictures: geometry in art, mathematics in nature, the use of triangles, optical illusions, geometric church windows, making multiplication easy, circles, geometry in the home, bridge builders as mathematicians, parallel lines, symmetry, mathematics in photography, vocations requiring a background of mathematics, objectives of mathematics. Bulletin board display.

Vitalizing Geometry with Visual Aids. Richard Drake and Donovan Johnson. Mathematics Teacher, Vol. 33, pp. 56–59, February, 1940.

Lists projects for mathematics classes, objectives for using visual material, and equipment for the mathematics classroom; pictures, lantern slides, motion pictures, film slides, stereopticon pictures, and three-dimensional pictures. Bibliography.

Vitalizing Geometry through Illustrative Material. Idella Waters. Mathematics Teacher, Vol. 28, pp. 101–110, February, 1935.

A unit in plane geometry on geometry in nature, art, and industry. Bibliography.

Uses of Cardboard. Ruth Lane. Mathematics Teacher, Vol. 18, pp. 239–241, April, 1925.

Problems for board work are printed on large charts of manila cardboard. Two printing sets are used in making the charts, which are numbered and filed.

In geometry students draw medians of triangle, and check the center of gravity by balancing triangle on a pencil point. Perspective figures are drawn in solid geometry for each theorem and models made. Flash cards used for review.

The Teaching of Geometry. Clara A. Hart. School Science and Mathematics, Vol. 5, pp. 649–652, November, 1905; pp. 717–725, December, 1905.

"Such work may be given as cutting a proper parallelogram along a diagonal and showing by superposition that the diagonal of a parallelogram divides it into two equal triangles. A frame made of four wires jointed at the ends to move easily will impress the fact that four sides do not determine a quadrilateral. Three wires similarly joined will show the case of the triangle. The ideas gained by this kind of work must be *used* to enlarge experience. The mind must be alert to apply facts as soon as acquired."

Popularizing Plane and Solid Geometry. Gertrude V. Pratt. Mathematics Teacher, Vol. 21, pp. 412–421, November, 1928.

Use of the bulletin board, pictures in the classroom, a field trip, style show, slide rule, art and architecture, models.

You Can Make Them. Clara O. Larson. Mathematics Teacher, Vol. 35, pp. 182–183, April, 1942.

Rigidity of geometric figures, the isosceles triangle as a device for bisecting an angle, parallel rulers and a simple transit are described, with accompanying illustrations.

Use of Models in Teaching Plane Geometry. Mannis Charosch. High Points, Vol. 14, pp. 42–44, February, 1932.

Little "Tinkering" in Geometry. W. Knapp. High Points, Vol. 17, pp. 16–20, April, 1935.

Mathematical Principles Applied to Mechanical Drawing. Emil E. Shattow. School Science and Mathematics, Vol. 36, pp. 890–896, November, 1936.

On centering a drawing: (1) graphical method, (2) computative method, (3) mechanical method which makes use of a proportional triangle—a mechanical device consisting of a 45° celluloid right triangle, the inside edge of the base of which has graduations of 1/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8, 1/9, 1/11. The triangle is provided with a celluloid handle which is pivoted by means of a rivet C.

Optional Topics in Plane Geometry. Clara O. Larson. Mathematics Teacher, Vol. 30, pp. 188–189, April, 1937.

Mentions using pieces from an erector set to illustrate plane geometry; also student applications of geometry to art, music, mechanical drawing, and advertising (geometric trade-marks).

Our Geometric Environment. E. E. Watson. School Science and Mathematics, Vol. 39, pp. 258–259, March, 1939.

Illustrations from astronomy, light, perspective, art, radial symmetry in flowers, fish, and snow crystals, and nautilus shell, the cyclone, suspension bridge, mineral crystals.

Geometry for Everyone. Kenneth S. Davis. Mathematics Teacher, Vol. 35, pp. 64–67, February, 1942.

A list of fifty illustrations in student's environment shows applications of fundamental facts of geometry.

A Number of Things for Beginners in Geometry. Vesta A. Richmond. Mathematics Teacher, Vol. 20, pp. 142–149, March, 1927.

Student interest focused on various activities each week: examples of mathematics in nature, construction of regular polyhedrons, simple transit instruments, measuring irregular pieces of land and drawing to scale, alum and salt crystals, crystallography, geometric designs, polygons from puzzles such as the tangram, models of locus theorems, proportion and the pantograph, dynamic symmetry.

Some Suggestions for Using Amateur Photography in Mathematics Courses. T. L. Engle. School Science and Mathematics, Vol. 33, pp. 506–510, May, 1933.

Suggests student with a camera can obtain pictures illustrating symmetry, circular and Gothic windows, compound curves, eccentric and concentric circles, proportion, similarity, triangular bracing, parabolas formed by suspended cables. For locus problems, time exposures may be taken at night of a point represented by a piece of burning magnesium or flashlight traveling around a wheel, a cycloid from a moving wheel, concentric circles shown by stars around north pole star. Students may make slides for classroom use and, using an ordinary camera, make stereoscopic pictures for three-dimensional impressions by snapping pictures of the same object in two positions, three inches apart.

Colored Crayons as an Aid in Teaching Mathematics. Ada M. Parsons. School Science and Mathematics, Vol. 14, pp. 33–35, January, 1914.

Only a few colored crayons are needed in proving the theorems on congruence of triangles, the proposition that angles opposite the equal sides of an isosceles triangle are equal, in locus problems, etc.

A Device to Aid in Generalizing Geometrical Idea. Henry Kemmerling. School Science and Mathematics, Vol. 27, pp. 606–609, June, 1927.

A board, 2½ feet square and ¾ inch thick, is perforated with ⅛ inch holes, 2 inches apart. No. 14 casing nails fit holes. Elastic bands are stretched between nails for straight lines.

A circle of variable size is made from an 8-foot saw with the teeth ground off. This strip bends into a nearly perfect circle and is held in place by small blocks.

Developing a Concept of Proportion before Presenting the Formal Work. T. L. Engle. School Science and Mathematics, Vol. 32, pp. 268–271, March, 1932.

Cartoons and pictures are enlarged with the aid of graph paper. Illustrated.

A Geometry Pupil's Brilliant Work. George W. Evans. School Science and Mathematics, Vol. 6, pp. 595–597, October, 1906.

A mediocre student can be led to do his own thinking in geometry to discover interesting geometric properties.

A Suggestion for Review in Geometry. C. A. Petterson. School Science and Mathematics, Vol. 7, p. 704, November, 1907.

Illustrates a composite figure to be mimeographed and used to discover geometric properties for review work. Other similar figures will suggest themselves.

Home-made or Inexpensive Mathematical Apparatus. Joseph V. Collins. School Science and Mathematics, Vol. 7, pp. 524–528, June, 1907.

Discusses and illustrates an algebraical balance, solid geometry modeling frame, plane table, pantagraph, blackboard compass-protractor, slide rules.

The Terminology of Elementary Geometry. Louis C. Karpinski and Adelaide M. Fiedler. School Science and Mathematics, Vol. 24, pp. 162–167, February, 1924.

A table shows the origin, from Greek or Latin, of terms used in geometry and the first English author to use the word. Material can be presented in a chart for classroom use or for school exhibit.

A Device for Teaching Locus. Margaret C. Amig. Mathematics Teacher, Vol. 34, p. 279, October, 1941.

Locus Makes a Plea. D. McLeod. Mathematics Teacher, Vol. 27, pp. 336–339, November, 1934.

Locophobia: Its Causes and Cure. George Selleck. Mathematics Teacher, Vol. 22, pp. 382–389, November, 1929.

Locus in plane and solid geometry: "A locus a day keeps phobia away."

Teaching the Locus Concept in Plane Geometry. E. B. Woodford. Mathematics Teacher, Vol. 26, pp. 105–106, February, 1933.

Describes materials used by students in visualizing the locus but not in constructing it. Discusses a number of experiments which can be carried out with these materials.

SOLID GEOMETRY

The Teaching of Solid Geometry at the University of Vermont. G. H. Nicholson. Mathematics Teacher, Vol. 30, pp. 326–330, November, 1937.

Most of the important solids in solid geometry were constructed and models were made of the main propositions. Desargue's Theorem and other theorems from modern pure solid geometry were also included. Two illustrations show models made of cartons, string, electric wire, knitting needles, umbrella stays, small thin pieces of wood, colored paper, toothpicks, cardboard, airplane toys, copper wires, tin, screening, a boy's building set, a salt container, and soldering wire. Fifteen objectives are discussed and fifteen references listed.

A Color Scheme for Figures in Solid Geometry. M. C. Spencer. School Science and Mathematics, Vol. 10, p. 549, June, 1910.

"(1) A color is used to outline the portion of each plane used; (2) all lines in a plane, provided they are not intersections of the plane with another, are drawn in the color chosen for outlining that plane; (3) all lines common to two or more planes are drawn white on the blackboard and black on paper."

Have You Tried Clay? Harriet B. Herbert. School Science and Mathematics, Vol. 42, pp. 218–220, March, 1942.

Clay and sticks are used in flat boxes or on spheres to help illustrate many of the theorems of solid geometry.

Solid Geometry Modeling Frame. C. E. Comstock. School Science and Mathematics, Vol. 4, p. 171, 1904.

Models in Solid Geometry. Miles C. Hartley. Mathematics Teacher, Vol. 35, pp. 5–7, January, 1942.

String models are made to help visualize theorems.

A Helpful Technique in Teaching Solid Geometry. James V. Bernardo. Mathematics Teacher, Vol. 33, pp. 39–40, January, 1940.

Models made of inexpensive materials—paper, sticks, string, cardboard, etc.

A Special Method in Mensurational Geometry. Sister Mary Gerard. School Science and Mathematics, Vol. 31, pp. 987–988, November, 1931.

A classroom experiment in which students used molding clay, protractor, and straightedge to mold a prism whose bases are squares and to find total area; then, with same amount of clay, to model a prism whose base was a rhombus.

Glass Models Show Third Dimension. J. E. Russell. Nation's Schools, Vol. 27, p. 29, April, 1941.

Modern Models for Modern Education. C. L. Connor. Bulletin of the Mathematics Section, Eastern Division, Colorado Education Association, Denver, Vol. 3, pp. 5–6, Spring, 1939.

Inverted Geometry. Daniel Luzon Morris. Mathematics Teacher, Vol. 31, pp. 78–80, February, 1938.

Models are used in introducing the work in geometry as well as in understanding some basic principles of descriptive geometry.

Models for Teaching Solid Geometry. School Science and Mathematics, Vol. 13, p. 210, March, 1913. Reprinted from The School World.

"Cut three pieces of 16 S.W.G. copper wire, each 2½ inches long; hold them together, and bind them round the middle with about ten turns of 22 S.W.G. copper wire. Touch this part with a little solder, and then cut into two equal lengths. Bending the ends of the wires outwards forms two little tripods. A supply of three- and four-legged pieces is required.

"From the toy shops or ironmongers, round wooden rods about 5/16 inch in diameter can be obtained, which are sold in 4-foot lengths. Cut them up into lengths varying by 1 inch, the largest 15 inches, the smallest 10 inches, and with a twist drill rather smaller than 16 S.W.G. make a hole at each end axially.

"By inserting the legs of the tripods, etc., into these holes any framework can quickly be built up.

"By rolling a piece of thin brass round one of the rods a split tube can be made with which rods can be joined telescopically and any length obtained."

The Tesseract, $(a + b)^4$, a Demonstration of the Binomial Theorem in Fourth Dimensional Geometry. Harriet B. Herbert. National Mathematics Magazine, Vol. 15, pp. 97–99, November, 1940.

A Study of the Cultivation of Space Imagery in Solid Geometry through the Use of Models. Edwin W. Schreiber. Mathematics Teacher, Vol. 16, pp. 102–111, February, 1923.

Discusses arguments against using models. Points out need for developing space imagery in student. Illustrates forty-one models and patterns for twelve of these. Bibliography.

Solid Geometry Made More Interesting. Marvin C. Volpel. School Science and Mathematics, Vol. 38, pp. 740–742, October, 1938.

A model of the Taj Mahal made of cardboard and ten boxes of matches; domes of plaster of Paris gilded.

The Regular Star Solids. Gertrude V. Pratt. School Science and Mathematics, Vol. 28, pp. 463–467, May, 1928.

A discussion of regular polyhedra, Archimedean solids, and four regular star polyhedra: the small stellated dodecahedron, the great dodecahedron, the great stellated dodecahedron, and the great icosahedron. Illustrated.

Construction Work in Solid Geometry. Edwin W. Schreiber. School Science and Mathematics, Vol. 19, pp. 407–413, May, 1919.

Based on Kepler's "To measure is to know." Students in solid geometry calculate dimensions for polyhedrons whose volume is ten cubic inches. Thirty models are illustrated, requirements for fifteen of these are listed.

The Rotostat and Goniostat: A Teacher's Classroom Device for Instruction in Projection. Herman Hanstein. School Science and Mathematics, Vol. 9, pp. 868–870, December, 1909.

A set of models in skeleton form with which the regular and an endless number of irregular surfaces, solids, and their penetrations are mounted before the class. The models may be rotated in vertical and horizontal revolutions.

Looking at Solid Geometry through Perspective. Ethel Spearman. Mathematics Teacher, Vol. 34, pp. 147–150, April, 1940.

Figures from solid geometry, including spherical figures, are drawn, using the basic principles of perspective.

Gaging a Horizontal Cylinder. William F. Riggs. School Science and Mathematics, Vol. 16, pp. 31–34, January, 1916.

Explains a diagram which may be used to obtain a vertical scale for the amount of liquid present at any time in a horizontal cylindrical tank.

Tennis Ball Geometry. H. F. Hart. Mathematics Teacher, Vol. 15, pp. 239–240, April, 1922.

Uses of a "dead" tennis ball illustrated in the solution of ten problems of solid geometry.

The Dandelin Spheres. Lee Emerson Boyer. Mathematics Teacher, Vol. 31, pp. 124–125, March, 1938.

Theorem: section of a cone of revolution made by plane which does not pass through the cone's vertex is an ellipse, parabola, or hyperbola. The foci of the sections are the points of contact of the intersecting plane with the spheres inscribed in the cone; the directrises of these sections lie in the plane determined by the circles of contact of the spheres with the cones in which they are inscribed. Bibliography.

Polar Triangles. William F. Riggs. School Science and Mathematics, Vol. 6, pp. 663–666, November, 1906.

Suggests making a spherical triangle from cardboard and fitting a corresponding polar triangle into it. Another substitute is a cheap six-inch terrestrial globe with pins stuck into it and connected by colored strings.

A Model of Supplementary Trihedral Angles. R. M. Mathews. School Science and Mathematics, Vol. 18, pp. 846–848, December, 1918.

The model consists of two main parts, one for each of the trihedral angles.

PLANE AND SPHERICAL TRIGONOMETRY

Visualizing Trigonometry. L. E. McAllister. School Science and Mathematics, Vol. 31, pp. 827–828, October, 1931; Vol. 32, pp. 895–900, November, 1932.

Drawn on a piece of white cardboard 1′ by 2′ tacked to a flat wooden board are a circle of 4″ radius, the *X* and *Y* axes, perpendicular tangents to the circle at the intersections of the circle with the positive *X* and *Y* axes. A heavy stiff wire 2′ long, ⅛″ in diameter rotates at the center of circle; another wire hangs freely from this wire at the radius distance from center. By rotating the first wire, values of sines and cosines may be determined. Variation of the trigonometric functions in the four quadrants may also be shown.

A Device for Teachers of Trigonometry. Norman Anning. School Science and Mathematics, Vol. 25, pp. 739–740, October, 1925.

Device consists of three bars which move in parallelogram fashion for noting variation of trigonometric functions of the general angle. Negative values are suggested in red. Scales graduated in tenths or hundredths may be pasted at necessary locations. The author queries: "Would there be any advantage in having each pupil make one for himself? How would he graduate the circles? Is this or some similar mechanism offered by the supply houses to assist physics teachers in treating simple harmonic motion?"

A Method of Demonstrating and Teaching the Trigonometric Functions. S. C. Mitchell. School Science and Mathematics, Vol. 17, pp. 245–247, March, 1917.

The "trigonometric function indicator" consists of a plane surface upon which is laid out a quadrant of a circle of unit radius. The cosine scale measured in tenths and hundredths is the horizontal scale. The tangent scale appears as a tangent to the circle perpendicular to the horizontal. The sine, secant, and cotangent scales are also shown.

An Instrument for Drawing a Sine Curve. A. Stanley Mackenzie. Physical Review, Vol. 15, pp. 366–367, 1908.

Describes construction of instrument, together with illustrations.

The Sine Function in Nature. Abraham M. Ginsburg. High Points, Vol. 19, pp. 69–70, November, 1937. Reviewed in Mathematics Teacher, Vol. 31, p. 35. January, 1938.

Discusses simple experiments from sound, mechanics, and electricity. Describes a crude device for drawing a part of a sine curve on paper or on the blackboard.

Cartographical Projections for Geographical Maps. Alexis M. Uzefovich. School Science and Mathematics, Vol. 38, pp. 378–390, April, 1938.

Conformal, equal-area, perspective, azimuthal, conical, and other projections are classified. Among the projections discussed and illustrated are Mollweide's, Mercator's, Lambert's, Goode's, and Bonne's.

Graphical Determination of the Distance between Two Given Points on or near the Surface of the Earth. Alexis M. Uzefovich. School Science and Mathematics, Vol. 28, pp. 951–958, December, 1928.

Using two cards, the writer shows how the distance between two points on the earth's surface can be approximated. Also discusses use of an "airline-distance meter" which is illustrated.

A Straight Line Chart for the Solution of Spherical Triangles. Journal of the Washington Academy of Sciences, No. 17, October 19, 1924.

The Graphical Solution of Spherical Triangles. Myron O. Tripp. School Science and Mathematics, Vol. 20, pp. 734–742, November, 1920.

Eleven diagrams show how scale drawings may be used to solve right spherical triangles.

JUNIOR COLLEGE MATHEMATICS

A Homemade Planimeter for Classroom Use. Ernest W. Ponzer. School Science and Mathematics, Vol. 11, pp. 242–245, March, 1911.

Although suggested for students in the calculus, the planimeter described could be used wherever congruent, equivalent, proportional, or irregular areas are studied. Illustrated.

How a Polar Planimeter Works. Allan W. Larson. School Science and Mathematics, Vol. 35, pp. 932–941, December, 1935.

Discusses general characteristics of the planimeter, theory of the motion of tracing arm, polar arm together with the tracing arm, and an application of the polar planimeter in calculating the indicated horsepower of a steam engine.

The Hatchet Planimeter. Herbert E. Cobb. School Science and Mathematics, Vol. 15, p. 802, December, 1915.

Gives directions for making the instrument from wire one-eighth inch in diameter. Can be readily made and used by any student of geometry.

Geometrical Representations of the Terms of Certain Series and Their Sums. Adrian Struyk. School Science and Mathematics, Vol. 37, pp. 202–208, February, 1937.

Using wooden blocks, the following are illustrated: the sum of the first *n* numbers, the sum of the first *n* odd numbers, the sum of products of $n(n+1)$, the sum of the squares of the first *n* numbers, the sum of the squares of the first *n* odd numbers, the sum of the cubes of the first *n* numbers. These blocks aid in developing the formulas for these simple series.

Graphical Representation of a Geometrical Series. D. H. Richert. School Science and Mathematics, Vol. 15, pp. 21–22, January, 1915.

A Mechanical Solution of the Cubic. Robert C. Yates. Mathematics Teacher, Vol. 32, p. 215, May, 1939.

A cardboard strip attached to the plane is used to determine solutions of the cubic equation.

The Use of Models While Teaching Triple Integration. E. A. Whitman. American Mathematical Monthly, Vol. 48, pp. 45–48, January, 1941.

Integraphs. Conrad K. Rizer. School Science and Mathematics, Vol. 33, pp. 1003–1005, December, 1933.

A brief description of eight integraphs. Reviews the literature on integraphs that have not been placed in the hands of manufacturers to market. In the United States there are integraphs at Cornell University and Massachusetts Institute of Technology. Bibliography.

GRAPH PAPER

The Cross-Section Paper as a Mathematical Instrument. Eliakim Hastings Moore. School Science and Mathematics, Vol. 6, pp. 429–450, June, 1906.

Discusses the systematic use of cross-section paper as a unifying element in mathematics. Illustrates double entry tables—numerical and graphical—and sets up devices or linkages for graphical computations.

Graphic Railroad Time-Tables. Florian Cajori. School Science and Mathematics, Vol. 10, pp. 204–205, March, 1910.

The course of every train is plotted by oblique lines. "The diagram shows at a glance when a train arrives at each station, also where the trains meet and pass each other. In railroad offices, the graphic time-charts are prepared first and the time-tables in ordinary use are copied from them."

Graph Tracing. F. C. Boon. School Science and Mathematics, Vol. 5, pp. 398–401, June, 1905.

Encourages tracing of curves rather than plotting of points.

An Experiment in the Use of Graph Papers. Norma Sleight. Mathematics Teacher, Vol. 35, pp. 84–87, February, 1942.

Uses made of various types of paper studied as part of class work.

Toward Better Graphs. Edwin Eagle. Mathematics Teacher, Vol. 35, pp. 127–131, March, 1942.

Nine important properties which graphs should possess.

A Practical Three-Dimensional Graph. Franklin Miller, Jr. School Science and Mathematics, Vol. 34, pp. 919–922, December, 1934.

"Contour sheets" are cut out of stiff cardboard sheets and interlocked with others. Slits should be wide enough, cuts staggered. Each sheet should be labeled and x and y co-ordinates marked along lower edge. Completed graph may be filled with plaster of Paris or paraffin, if a permanent model is desired.

A Use for the Graph in Elementary Chemistry. Charles H. Stone. School Science and Mathematics, Vol. 36, pp. 281–283, March, 1936.

Two chemistry problems solved graphically. Lists eight typical problems for graphical solution.

See also BOOKS:

Enriched Teaching of Mathematics. M. Woodring and V. Sanford. Pp. 52–53.

THE SLIDE RULE

A Student-Made Slide Rule. William E. Bush. Mathematics Teacher, Vol. 27, pp. 405–408, December, 1934.

Uses semi-logarithmic paper strips about 1/4" wide. Paste is applied to cardboard, and strips are pasted onto it. The cardboard slider is made from an old piece of photographic film. Rule is sufficiently accurate to guarantee two digits.

The Slide Rule Constructed without Logarithms. A. H. Fensholt. School Science and Mathematics, Vol. 15, pp. 417–421, May, 1915.

Using a unit for the length of the slide rule D scale from 1 to 2, the points 4, 8, 16 are laid off. By means of a graph units for 1 to 3, 1 to 5, 1 to 7, etc., are determined. This helps students to understand the subdivisions of the slide rule but does not give an accurate scale.

A Slide Rule for Classroom Use. Ernest W. Ponzer. School Science and Mathematics, Vol. 10, pp. 776–779, December, 1910.

Directions for making a classroom slide rule are given. "In this day we do not hesitate to use, in fact it is our duty to use, all those aids which will help round out any course in mathematics from arithmetic up." Illustrated.

Construction of a Demonstration Slide Rule. Earl C. Rex. School Science and Mathematics, Vol. 40, pp. 161–164, February, 1940.

Using student assistance, a 7′ x 9″ x 2″ ruler for the classroom is made at a cost of about a dollar.

The Slide Rule as a Subject of Regular Class Instruction in Mathematics. William E. Breckenridge. Mathematics Teacher, Vol. 14, pp. 342–343, October, 1921.

Advocates teaching the use of slide rule early in the mathematics course.

The Slide Rule in Junior High School. Charles A. Stone. School Science and Mathematics, Vol. 30, pp. 645–650, June, 1930.

Lists topics for eighteen meetings of junior high school slide rule class.

A Slide Rule in Junior and Senior High School. John F. Barnhill. Mathematics Teacher, Vol. 17, pp. 359–364, October, 1924.

Advocates the use of the slide rule in seventh and eighth grade, especially in checking calculations. References.

The Slide Rule in Plane Geometry. W. W. Gorsline. Mathematics Teacher, Vol. 17, pp. 385–403, November, 1924.

A detailed account of how the slide rule can be used in solving fifty-seven problems relating to twenty-two theorems of plane geometry.

The Slide Rule as a Check in Trigonometry. William E. Breckenridge. Mathematics Teacher, Vol. 23, pp. 52–59, January, 1930.

Multiplication and division; proportion and simple square root; checking right and oblique triangles using the law of sines.

The Slide Rule in the Junior College. W. W. Gorsline. Mathematics Teacher, Vol. 26, pp. 292–295, May, 1933.

Outline for twenty lessons: reading of the slide rule, multiplication using C and D scales, division using C and D scales, proportion, combined multiplication and division, and additional topics.

Adapting the Slide Rule to High School Chemistry. Hoyt C. Graham and John Huff. School Science and Mathematics, Vol. 30, pp. 525–528, June, 1930.

Outlines procedures to be followed in teaching slide rule to a high school chemistry class.

The Slide Rule in Business. S. L. Shelley. Mathematics Teacher, Vol. 14, pp. 261–263, May, 1921.

"If I were to interpret for you the attitude of the average business man towards mathematics today, I believe that it would be something as follows: 'Teach as much mathematics as you can; the more the better. Get the following practical things into your course as early as you possibly can, certainly before the student has finished high school: (1) teach him to add; (2) teach him to subtract; (3)

teach him to multiply; (4) teach him to divide; (5) teach him simple equations; (6) teach him measurement of all kinds; (7) teach him charting, so that he can both understand and make charts; (8) teach him the slide rule, that he may save his own time and mine in the numberless calculations that come up in the transaction of business'."

Helping the Student Choose a Slide Rule. B. Cecil. Bulletin of the Kansas Association of Teachers of Mathematics, Vol. 15, p. 68, April, 1941.

Time Is Money. William E. Breckenridge. Mathematics Teacher, Vol. 16, pp. 332–334, October, 1923.

"A story for high school and college students and for all who are interested in saving time." One boy tells another of uses and method of using slide rule.

A Convenient Rule for Locating the Decimal Point in Slide Rule Calculations. Lloyd C. Elliott. School Science and Mathematics, Vol. 26, pp. 957–959, December, 1926.

New Method for Decimal Point on the Slide Rule. F. H. Wade. School Science and Mathematics, Vol. 32, p. 381, April, 1932.

The Slide Rule Watch. Robert L. Burg and Walter V. Burg. School Science and Mathematics, Vol. 42, pp. 72–74, January, 1942.

A device consisting of a dial and a pointer which may be used for locating decimal point in slide rule calculations.

Talk on Logarithms and Slide Rules. Florian Cajori. School Science and Mathematics, Vol. 20, pp. 527–530, June, 1920.

Illustrates title page of the earliest book published on the slide rule.

Solution of Quadratic and Cubic Equations on the Slide Rule. R. C. Colwell. Mathematics Teacher, Vol. 19, pp. 162–165, March, 1926.

Slide Rule Solutions of Quadratic and Cubic Equations. T. J. Higgins. American Mathematical Monthly, Vol. 44, pp. 646–647, December, 1937.

The Slide Rule in the Solution of Cubic Equations. L. E. Curfman. Bulletin of the Kansas Association of Teachers of Mathematics, Vol. 11, pp. 4–5, October, 1936.

Paper based on article by R. A. Whiteman in Civil Engineering, October, 1934.

The Right-Angle Slide Rule. Oscar G. Fryer. School Science and Mathematics, Vol. 36, pp. 422–425, April, 1936.

Specifications of a right-angle slide rule for finding all trigonometric functions and for solving any problem involving the relation of the sides of a triangle. Also applications to college physics. Dimensions, materials, etc. are listed.

A Slide Rule for Quadratic Equations. I. C. Barker. School Science and Mathematics, Vol. 35, pp. 811–813, November, 1935.

Shows how the scales for such a slide rule may be constructed and used to find the roots of a quadratic equation, whether it is factorable or not.

See also BOOKS:

Enriched Teaching of Mathematics. M. Woodring and V. Sanford. Pp. 54–55.

CALCULATING INSTRUMENTS

An Improved Abacus. Stella E. Myers. School Science and Mathematics, Vol. 7, pp. 601–603, October, 1907.

Describes an instrument consisting of beads on curved wires.

Some Mathematics of the Calculating Machine. L. Leland Locke. Mathematics Teacher, Vol. 15, pp. 423–428, November, 1922.

An introduction to the operation and understanding of this machine.

Mathematics of the Calculating Machine. L. Leland Locke. Mathematics Teacher, Vol. 17, pp. 78–86, February, 1924.

Discusses automatic operations on the calculating machine.

The Use of Calculating Machines in Teaching Arithmetic. [illegible] S. Schlauch. Mathematics Teacher, Vol. 33, pp. 35–38, January, 1940.

Calculating Machines and the Mathematics Teachers. Evelyn M. Horton. Mathematics Teacher, Vol. 30, pp. 271–276, October, 1937.

Discusses the listing machine, bookkeeping machine, key-punch and tabulating machine, and key-driven and non-key-driven calculating machine.

Abacus. David Eugene Smith. *Encyclopædia Britannica,* 14th Edition, 1939, Vol. 1, pp. 603–607.

Calculating Machines. David Baxandall. *Encyclopædia Britannica,* 14th Edition, 1939, Vol. 4, pp. 548–553.

SURVEYING INSTRUMENTS

Constructing a Transit as a Project in Geometry. T. L. Engle. Mathematics Teacher, Vol. 24, pp. 444–447, November, 1931.

"The only materials that need to be purchased are two protractors, a pocket compass, and three small unmounted spirit levels." Dimensions and directions for constructing a transit for approximately two dollars. Illustrated.

A Few Live Projects in High School Mathematics. Frank M. Rich. School Science and Mathematics, Vol. 20, pp. 34–45, January, 1920.

Directions for making a chromatic zither from a box of thin wood or from tin, a samisen or type of banjo from a cigar box, a xylophone from bars of wood or lengths of pipe, a homemade transit (of interest to boys and girls) and homemade stadia. All materials are listed.

A Project in Navigation. Edna Carlson. School Science and Mathematics, Vol. 36, pp. 580–584, June, 1936.

Fundamentals of the use of the sextant with four simple devices for illustration. (Construction of actual sextant given in Popular Science Monthly, p. 34, September, 1933.) Use of the sextant; sample observations. An old umbrella is made into a device for showing the relationship of the sun to the celestial equator. References.

A Device for the Teaching of the Operation of a Sextant. John B. Leake. School Science and Mathematics, Vol. 35, pp. 923–924, December, 1935.

Instructions and diagrams for constructing and using a sextant.

I Can't See Geometry. George A. Boyce. Educational Screen, Vol. 12, pp. 40–41, February, 1933.

Ancient, medieval, and modern surveying instruments are studied and used by students. They also make model scenes to illustrate textbook problems. Use of colored chalk, student's cartoons, rotating blackboard, and a mathematics laboratory are also discussed.

An Experiment in Field Mathematics. Sara Belle Wasser. Bulletin of the Kansas Association of Teachers of Mathematics, Vol. 13, pp. 7–8, February, 1939.

Equipment used, units studied, benefits derived from course.

Map Construction. William H. Snyder. School Science and Mathematics, Vol. 5, pp. 29–38, January, 1905.

Discusses laboratory work in map construction.

Instruments for Topographic Surveying. Willard S. Bass. School Science and Mathematics, Vol. 5, pp. 167–172, March, 1905.

Discusses tripod, plane table, alidade, flagpoles, measuring tape, pins, level, leveling rod, and stadia and stadia rod needed for group work.

Early Forms of a Few Common Instruments. William E. Stark. School Science and Mathematics, Vol. 9, pp. 871–874, December, 1909; Vol. 10, pp. 48–67, 126–139, January, 1910.

Illustrates parallel rulers dated 1686 and 1723, a pantograph used in copying pictures in 1645, and a semi-circular slide rule of 1696, instruments for measuring distances and angles, the geometric square, the cross-staff, sector compasses, map-making instruments, leveling instruments, and artillerist's instruments. Thirty-eight figures, many of which could be reproduced by students.

The Cross-Staff. W. E. Schimpff. Mathematics Teacher, Vol. 24, pp. 320–321, November, 1941.

Describes its construction and discusses problems which it helps to solve. References.

Plane Table and Accompanying Apparatus. Willard S. Bass. School Science and Mathematics, Vol. 4, p. 207, March, 1904.

A Simple Hypsometer. E. N. Trauseau. School Science and Mathematics, Vol. 7, pp. 114–117, February, 1907.

Directions for constructing a simple hypsometer and its use in finding heights of trees and in topographic surveying.

See also BOOKS:

Field Work in Mathematics. Carl N. Shuster and Fred L. Bedford.

Mathematics of Everyday Life, Geometry Unit. George A. Boyce and W. W. Beatty. Pp. 4–7 (the range-finder), 37–41 (the transit), 49–52 (the cross-staff), 60–63 (drumhead surveying and map making), 101–116 (staff and range finder and isosceles triangle), 148–156 (quadrant and sun dial).

Surveying for Schools and Scouts. W. A. Richardson. Pp. 34–38, 47–49, 89–94.

Enriched Teaching of Mathematics. M. Woodring and V. Sanford. Pp. 46–50.

ALIGNMENT CHARTS

Alignment Charts. Joseph Lipka. Mathematics Teacher, Vol. 14, pp. 171–178, April, 1921.

Solutions of problems taken from industry, percentage problems, finding graphically the roots of any quadratic equation. Building a graphical chart for multiplication and division. References.

How to Figure Averages with the Top of a Shoe Box, or the Use of an Alignment Chart in Averaging. J. A. Roberts. Mathematics Teacher, Vol. 17, pp. 471–474, December, 1924.

Nomography. J. S. Georges and W. W. Gorsline. School Science and Mathematics, Vol. 36, pp. 267–272, March, 1936.

On the graphic solution of equations using nomography (parallel lines). Nomogram of (1) $x + y = c$; (2) $x - y = z$; (3) $z = ax + by$; (4) $z = x - y + w$; (5) $z = xy$; (6) $a = \frac{32w}{cdv^2}$.

Value of Nomographic Charts in the Teaching of Physics. Albert Coyne. School Science and Mathematics, Vol. 37, pp. 720–724, June, 1937.

Nomography. M. d'Ocagne. Encyclopaedia Britannica, 14th Edition, 1939, Vol. 16, pp. 483–484.

THE SUN DIAL

The Mathematics of the Sun Dial. LaVergne Wood and Frances Mack Lewis. Mathematics Teacher, Vol. 29, pp. 295–303, October, 1935.

A unit on the sun dial based on solid geometry and spherical trigonometry. Theory and construction of the horizontal sun dial, and setting the dial, are included. References.

The Sun-Path Dial. Joseph F. Morse. School Science and Mathematics, Vol. 8, p. 561, October, 1908.

An adjustable device for demonstrating the course of the sun above and below the horizon of different latitudes at different times of the year.

A Method of Finding the Meridian by Shadows and Mechanically Graduating a Sun Dial. Edison Pettit. School Science and Mathematics, Vol. 10, pp. 483–486, June, 1910.

Five figures show how to mark the sun dial.

An Adjustable Sun Dial. J. F. Morse. School Science and Mathematics, Vol. 15, pp. 740–741, November, 1915.

Describes an invention of Mr. Leinert on which necessary adjustments are made for latitudes 30° to 45° N.

HOMEMADE INSTRUMENTS FOR DRAWING CURVES

Blackboard Compass. Perry Ross. School Science and Mathematics, Vol. 25, p. 542, May, 1925.

On one side of a wooden slat, ½" thick and 1¼" wide, are cut a series of V notches. A crayon is held in place by a rubber band. The device may be made in a few minutes and is serviceable where an inexpensive compass is wanted for blackboard use.

The Conic Compass. John L. C. Löf. School Science and Mathematics, Vol. 38, pp. 842–846, November, 1938.

Idea of this compass based on that of the elliptic compass (Mathematical Dictionary, Davies and Peck, 1875, p. 112). Point moves on the intersection of a right circular cone and a plane. Mechanical construction, theory, and operation are explained.

A Simple Blackboard Ellipsograph. Haym Kruglak. Mathematics Teacher, Vol. 33, p. 179, April, 1940.

Two diagrams and directions for construction of instrument which requires inexpensive suction cups mounted on a sheet metal frame.

An Ellipsograph. Robert C. Yates. National Mathematics Magazine, Vol. 12, pp. 213–215, February, 1938.

A linkage for drawing ellipse is explained and illustrated.

A New Ellipsograph. J. A. VanGroos. School Science and Mathematics, Vol. 22, pp. 471–472, May, 1922.

Two arms are made to rotate about a point. The middle point of the line joining the ends of the arms will generate an ellipse. Suggested by watching the opening and closing of gates at a railroad crossing a city street.

Ellipsographs. David Baxandall. Encyclopaedia Britannica, 14th Edition, 1939, Vol. 8, pp. 370–371.

LINKAGES

How to Draw a Straight Line. F. E. Tuck. School Science and Mathematics, Vol. 21, pp. 554–558, June, 1921.

Ten linkages are described. These models are very suggestive scientific toys for mechanically minded boys.

Linkages. Joseph Hilsenrath. Mathematics Teacher, Vol. 30, pp. 277–284, October, 1937.

Watt's approximate straight-line motion; Tschebicheff's and Robert's approximate straight-line-motion linkages; Peaucellier's cell and Bricard's exact straight-line motion; Peaucellier's and Robert's conicographs; and Kempe's angle trisector, discussed and illustrated.

The Story of the Parallelogram. R. C. Yates. Mathematics Teacher, Vol. 33, pp. 301–310, November, 1940.

Parallelograms are dissected with straight lines to form a triangle and a square. If the triangles formed by joining the mid-points of the sides of any quadrilateral are rearranged, they will coincide with the remaining parallelogram. Parallelograms in motion are illustrated with various linkage systems. The pantograph and lazy tongs are shown to be illustrations. Other linkages are also explained. Bibliography.

Linkages. R. L. Hippesley. Encyclopaedia Britannica, 14th Edition, 1939, Vol. 14, pp. 163–164.

TELESCOPES

Telescope-Making in the West Allis High School. Harold R. Stamm. School Science and Mathematics, Vol. 37, pp. 643–650, June, 1937.

Directions are given on how to grind the disk and to make a telescope mounting. Inexpensive materials.

Amateur Telescope-Making. Willard Geer. School Science and Mathematics, Vol. 34, pp. 76–79, January, 1934.

A Skeleton Telescope. E. C. Woodruff. School Science and Mathematics, Vol. 2, pp. 340–341, 1902.

Describes and illustrates a simple working model.

Amateur Telescope-Making. James L. Russell. School Science and Mathematics, Vol. 41, pp. 63–68, January, 1941.

Recommends that beginner have an inexpensive telescope-making kit. Discusses the mirror, grinding, polishing, the Foucault test, parabolizing, aluminizing, the diagonal, the cell, the tube, the eye piece, the mounting and adjusting. Refers to Amateur Telescope-Making, Scientific American Publishing Company, New York, as the telescope-maker's "Bible."

Practical Advice and Aid in Telescope-Making. Henry L. Yeagley. Sky Magazine, Vol. 2, pp. 18 ff., March, 1938; pp. 18 ff., April, 1938; pp. 14 ff., June, 1938; pp. 10 ff., May, 1938.

Homemade Telescopes Open the Heavens to Amateurs. Clifford L. Butcher. Sky Magazine, Vol. 2, pp. 12 ff., June, 1938.

EXPERIMENTS

To Have and to Hold. Robert C. Yates. National Mathematics Magazine, Vol. 14, pp. 2–4, October, 1939.

Discusses projects, devices, and constructions to motivate high school mathematics instruction.

The Efficiency of Certain Shapes in Nature and Technology. A suggested unit of instruction on intuitive solid geometry. May Hickey. Mathematics Teacher, Vol. 32, pp. 129–133, March, 1939.

Experiments which can be carried out in finding the maximum area for a given perimeter, shape of cylindrical tin can of one-quart capacity with least amount of tin, minimal surfaces of various soap films, and other activities for an eighth grade course. Thirteen references.

A Thread of Mathematical History and Some Lessons. Zoe Ferguson. School Science and Mathematics, Vol. 24, pp. 37–45, January, 1924.

Discusses experiments found in C. V. Boy's Soap Bubbles and the Forces Which Mold Them, which result in hyperboloids of one sheet. Parabolas are illustrated by capillary action. Eleven figures.

Soap Film Experiments with Minimal Surfaces. Richard Courant. American Mathematical Monthly, Vol. 47, pp. 167–174, March, 1940.

Experiments with wire frames and soap solution. Includes formula for making a satisfactory soap film solution.

Some of Nature's Curves. Conic Sections. Sue Avis Blake. School Science and Mathematics, Vol. 36, pp. 245–249, March, 1936.

A discussion of conics as related to astronomy.

Some of Nature's Curves. Simple Periodic Motion. Sue Avis Blake. School Science and Mathematics, Vol. 36, pp. 486–489, May, 1936.

Simple periodic motion produced by falling bodies and by a freely swinging mass viewed from various angles and acted upon by various forces. Brief reference to astronomy.

Some of Nature's Curves. Sue Avis Blake. School Science and Mathematics, Vol. 36, pp. 717–721, October, 1936.

The circle and sphere in nature; parabolas formed by projectiles; photographs of a stream from a hose forming a parabola. Catenaries treated briefly.

An Optical Illusion of Conic Sections. Galen Wood Ewing. School Science and Mathematics, Vol. 38, pp. 276–277, March, 1938.

Adjusting light of a flashlight on the wall to different positions results in circle, ellipse, parabola, and hyperbola. Illustrated.

Conic Sections Formed by Some Elements of a Plane Triangle. Aaron Bakst. Mathematics Teacher, Vol. 24, pp. 28–31, January 1931.

Given a triangle, the base of which is fixed in magnitude and position and the area also given, construct on that base all triangles having the same area: then the loci of the points of intersection of medians, perpendiculars constructed at mid-points of the sides, bisectors of angles, and altitudes of all triangles having the same area on the same base are respectively a straight line, a straight line, an ellipse, and a hyperbola.

A Simple Device to Illustrate the Path of Projectiles. John W. Scoville. School Science and Mathematics, Vol. 12, p. 194, March, 1912.

Small balls (beads or buck shot) are suspended from a yardstick at five-inch intervals by pieces of thread whose ratios in length to the first are as 1 : 4 : 9 : 16 : etc. The rod may be placed in any desired position, and the balls will hang in the path of projectile.

Note on a Device to Illustrate the Path of a Projectile. J. H. C. Bagby. School Science and Mathematics, Vol. 13, pp. 224–225, March, 1913.

To the device described by J. W. Scoville (see above) is attached a grooved track. A polished steel ball rolled down the track will follow the path shown by the balls suspended from the rod.

A Projectile Experiment. Oscar L. Starr. School Science and Mathematics, Vol. 33, pp. 956–958, December, 1933.

An experiment using a stream of water to show projectile's path.

A Short Foucault Pendulum. L. S. Weltz and L. D. Strong. School Science and Mathematics, Vol. 28, pp. 255–263, March, 1928.

Discusses theory of this pendulum and gives directions on how to make a working model with a suspension wire a little over nine feet in length and a forty-pound cannon ball for a bob. Eight figures.

Compound Harmonic Motion Apparatus. Cyril Ruhlmann. School Science and Mathematics, Vol. 12, pp. 767–771, December, 1912.

Describes a simple apparatus for recording Lissajous' figures.

Spark Recording of Lissajous' Figures in the Elementary Physics Laboratory. R. L. Edwards. School Science and Mathematics, Vol. 30, pp. 909–910, November, 1930. Shows advantages of spark recorded figures over those usually traced with sand.

A Demonstration Apparatus for the Composition of Two Simple Harmonic Curves. E. G. Plasterer. School Science and Mathematics, Vol. 34, pp. 424–426, April, 1934.

Pictures a homemade piece of apparatus to demonstrate the composition of two simple harmonic motions two ways—at right angles and at parallels. Two pictures show the apparatus and the curves obtained from various experiments.

Mechanical Devices for Drawing Lissajous' Figures. R. C. Colwell. School Science and Mathematics, Vol. 36, pp. 1005–1006, December, 1936.

Descriptions and diagrams of machines for blackboard work for obtaining the sine and cosine curve.

Walking and Climbing Tops. R. C. Colwell. School Science and Mathematics, Vol. 27, pp. 137–139, February, 1927.

Simple experiments with the gyroscopic top.

An Old Form of Top Revived. Charles T. Knipp. School Science and Mathematics, Vol. 20, pp. 113–116, February, 1920.

Describes a top made by a countryside cabinet-maker in the nineteenth century. Of interest to those acquainted with material in the books Spinning Tops by John Perry and Spinning Tops and Gyroscopic Motion by Harold Crabtree (Longmans, Green and Company, New York).

TRISECTION

The Trisector of Amadori. Marian E. Daniels. Mathematics Teacher, Vol. 33, pp. 80–81, February, 1940.

Description, theory, and two illustrations.

Why It Is Impossible to Trisect an Angle or Construct a Regular Polygon of Seven or Nine Sides with Ruler and Compass. Leonard Dickson. Mathematics Teacher, Vol. 14, pp. 217–223, May, 1921.

To Trisect an Acute Angle. C. S. Floyd. School Science and Mathematics, Vol. 6, pp. 358–359, May, 1906. Also see note on p. 526 of Vol. 6.

Construction made by a high school boy. It contains errors which may be discovered by alert students.

Bieberbach's Trisection Method. Raymond Garver. Scripta Mathematica, Vol. 3, pp. 251–255, July, 1935.

In addition to straight edge and compasses, requires only one construction with a movable right angle. Bibliography.

The Trisection of an Angle. W. B. G[illegible]ens. American Mathematical Monthly, Vol. 44, pp. 459-461, August-September, [illegible]37.

Trisection by means of an auxiliary curve: the "Fokale." References.

Trisecting Any Angle by Means of a Hyperbola. W. A. Knight. School Science and Mathematics, Vol. 10, pp. 582-583, October, 1910.

Another Trisection Fallacy. Robert E. Moritz. School Science and Mathematics, Vol. 6, pp. 621-622, October, 1906.

Discusses errors in article by C. S. Floyd, see above.

The Trisection of an Angle by Means of a Graduated Ruler and Compasses. Clarence Ohlendorf. School Science and Mathematics, Vol. 13, p. 546, June, 1913.

Graphical Trisection of an Angle. E. D. Pickering. School Science and Mathematics, Vol. 22, p. 548, June, 1922.

A Machine for Trisecting Angles. Alfred H. Thiessen. School Science and Mathematics, Vol. 14, p. 236, March, 1914.

Line Motion and Trisection. Robert C. Yates. National Mathematics Magazine, Vol. 13, pp. 63-66, November, 1938.

A linkage for trisection of angle as well as bibliography and three diagrams.

The Trisection Problem. Robert C. Yates. National Mathematics Magazine, I: Vol. 15, pp. 129-142, December, 1940; II: Vol. 15, pp. 191-202, January, 1941; III. Vol. 15, pp. 278-293, March, 1941; IV. Vol. 16, pp. 20-28, October, 1941; V. Vol. 16, pp. 171-182, January, 1942.

I and II: Trisection using the following curves: the quadratrix, the conchoid, the hyperbola, and the cycloid of Ceva. III: Fifteen mechanical trisectors are described.

A Rose Linkage, Trisection, and *the Regular Heptagon.* Robert C. Yates. School Science and Mathematics, Vol. 39, pp. 870-872, December, 1939.

The linkage traces a four-leaved rose and acts as an angle trisector.

The Trisection Problem. James H. Weaver. School Science and Mathematics. Vol. 15, pp. 590-595, October, 1915.

Discusses the solution of Archimedes and one of Pappus.

See also BOOKS:

Plane Geometry. Mirick, Newell, and Harper. Pp. 142-143.

Trisection. Robert C. Yates.

PAPER FOLDING

Paper Folding in Plane Geometry. Sarah L. Britton. Mathematics Teacher, Vol. 32, pp. 227-228, May, 1939.

How to determine the perpendicular bisector of a segment and the center of a circle whose arc is given.

Tieing a Strip of Paper into a Knot to Form a Pentagon. School Science and Mathematics, Vol. 26, p. 654, June, 1926.

Proof that the regular pentagon is formed.

Construction of Conic Sections by Paper Folding. Alfred J. Lotka. School Science and Mathematics, Vol. 7, pp. 595-597, October, 1907.

See also BOOKS:

Geometrical Exercises in Paper Folding. T. S. Row.

Tools. R. C. Yates. Pp. 54-65.

Winter Nights Entertainments. R. M. Abraham. Pp. 23-48.

Fun with Paper. J. Leeming.

Houdini's Paper Magic. Houdini. Pp. 117–140.
Procédés Originaux de Constructions Géométriques. E. Fourrey. Pp. 115–139.

DISSECTION

Addition by Dissection. Robert C. Yates. School Science and Mathematics, Vol. 40, pp. 801–807, December, 1940.

Suggests constructing cardboard models (colored poster board—about 14-ply—is recommended) to dissect geometric figures and rearrange parts into others: transformation of a given triangle to one having a specified shape; transformation of a given parallelogram to another having a specified shape; transformation of a quadrilateral to a triangle. References.

See also BOOKS:

Mathematical Recreations and Essays. Ball and Coxeter. Pp. 87–94, 106.
A Companion to Elementary School Mathematics. Boon. Pp. 100–111.
The Pythagorean Proposition. Loomis. Many figures used in the geometric proofs may be shown readily by dissection.
Introduction to Geometry. Siddons and Snell. Appendix. Shows Pythagorean Theorem in colored sections.

DUPLICATION OF THE CUBE

An Approximate Solution of the Problem of the Duplication of the Cube. A. A. Dmitrovsky. School Science and Mathematics, Vol. 13, pp. 311–312, April, 1913.

Describes a construction which gives the edge of a double cube with 1/12500 as the degree of accuracy.

The Derivation and Applications of the Conchoid of Nicomedes and the Cissoid of Diocles. Harry Roeser. School Science and Mathematics, Vol. 14, pp. 790–796, December, 1914.

A translation from Cantor's Lectures in the History of Mathematics, Vol. I.

Pappus's Solution of the Duplication Problem. James H. Weaver. School Science and Mathematics, Vol. 13, pp. 216–217, March, 1913.

A circle drawn on a round table is used as the basis for the solution.

USE OF GEOMETRICAL TOOLS

Constructions with a Double-Edged Ruler. R. C. Archibald. American Mathematical Monthly, Vol. 25, pp. 358–360, October, 1918.

Geometric Constructions. W. W. Beman. School Science and Mathematics, Vol. 10, pp. 528–529, June, 1910.

Five constructions with straightedge alone, eight with compasses alone, four with ruler alone, five with square alone.

Compass Geometry. Walter H. Carnahan. School Science and Mathematics, Vol. 32, pp. 384–390, April, 1932.

Using compasses alone: how to construct the points of intersection of a straight line and a circle, and how to construct the point of intersection of two straight lines. Also the history of the problem.

Geometrical Constructions without the Compasses. Walter H. Carnahan. School Science and Mathematics, Vol. 36, pp. 182–189, February, 1936.

Constructions which are possible and impossible, using only a straightedge and sect carrier. Discarding the compasses, five constructions are outlined using the straightedge alone.

Constructions with Limited Means. Marion E. Stark. American Mathematical Monthly, Vol. 48, pp. 475–479, August–September, 1941.

The Angle Ruler, the Marked Ruler, and the Carpenter's Square. Robert C. Yates. National Mathematics Magazine, Vol. 15, pp. 61–73, November, 1940.

Theory of these instruments is considered. The angle ruler and marked square are shown to be related by the principle of duality. References.

All Geometrical Construction May Be Made with Compasses. Michael Goldberg. School Science and Mathematics, Vol. 25, pp. 961–965, December, 1925.

See also BOOKS:

Ruler and Compasses. Hilda P. Hudson. London, Longmans Green and Company. 1916.

An Introduction to Modern Geometry. L. S. Shively. New York, John Wiley and Sons. 1939. Pp. 132–135.

Tools. Robert C. Yates. Pp. 6–33, 42–53, 66–81, 102–117.

Procédés Originaux de Constructions Géométriques. E. Fourrey. Pp. 47–59, 74–94.

THE MATHEMATICS LABORATORY

The Teaching of Algebra by the Laboratory Method. Charles W. Newhall. School Science and Mathematics, Vol. 5, pp. 40–45, January, 1905.

"Often the student can use to advantage a mode of attack similar to that used in the solution of construction problems in geometry, imagine the problem solved, and try to discover what steps were necessary to attain that end. . . . To make a more complete application of the methods of the laboratory to the teaching of algebra, we might make use of some of the devices and arrangements before referred to as externals, which are peculiar to the laboratory studies."

Some Experiences in Laboratory Mathematics and Their Results. Franklin Turner Jones. School Science and Mathematics, Vol. 5, pp. 406–410, June, 1905.

Discusses experiments tried in geometry: comparison of the inch and centimeter, laws of reflection, center of gravity of a triangle, law of refraction, the Jolly Balance to illustrate proportion (see School Science and Mathematics, March, 1904).

"As to the introduction of the subject of geometry the method of observation and experiment can, and I believe should, be extensively used. To put off its introduction until the first or second year of high school is, however, too late to realize the most reliable results."

Mathematical Laboratories. Karl Goldziher. School Science and Mathematics, Vol. 8, pp. 753–757, December, 1908.

"1. Above all we assert that the practical exercises should be begun in the lower classes and continued throughout the course.

"2. The mathematical laboratory ought to be independent of the physics and chemistry laboratories, and ought to contain all the models and apparatus necessary for instruction in measuring and weighing. In many cases the pupils can themselves make the required apparatus.

"3. In connection with arithmetic there should be work in collecting statistics, and extended computations; the necessary tables should be made out by the pupils themselves.

"4. In the interest of instruction in the upper classes, care must be taken that the student secures for himself the material for computation, by measurement and observation in mechanics, physics, chemistry, meteorology, surveying, and astronomy. . . .

"5. The accurate construction of graphs might be considered an important problem of the mathematical laboratory. . . .

"6. The preparation of geometrical models is important. . . .

"7. In the teachers' library special attention should be given to the literature of applied mathematics. . . ."

Experimental Geometry. H. J. Chase. School Science and Mathematics, Vol. 8, pp. 577–579, October, 1908.

Mathematics. William T. Campbell. School Science and Mathematics, Vol. 9, pp. 385–387, April, 1909.

"Indications seem to point to the need of a more practical basis for both algebra and geometry than we are now attempting—perhaps to a kind of mathematical laboratory. . . . If, however, the mathematical laboratory comes, let it come, not as an addition to what we are now doing, but as a substitute for parts of it. . . ."

Mathematical Laboratory. School Science and Mathematics, Vol. 13, pp. 544–545, June, 1913.

In October, 1913, University of Edinburgh opened a laboratory for practical instruction in numerical, graphical, and mechanical calculation and analysis, as required in the applied mathematical sciences, and for research in connection with the mathematical department. The practical work in the laboratory involves such topics as curve fitting; construction of curves and surfaces; linkages; roulettes; projections; photogrammetry; map-making; use of instruments employed in calculation, especially slide rules, arithmometers, planimeters, integraphs, and harmonic analysers; graphic solution of numerical equations; graphic and mechanical solution of problems in spherical trigonometry; nomography; applications of triangles of vectors; construction of tables of new functions and functions not previously tabulated.

A Dream Come True. William A. Austin. School Science and Mathematics, Vol. 21, pp. 621–627, October, 1921.

Describes classrooms which have been equipped as laboratories for plane geometry, the laboratory procedures used, and the quantity of work accomplished. Illustrated.

Experimental Geometry. G. A. Harper. Mathematics Teacher, Vol. 15, pp. 157–163, March, 1922.

Advocates the use of a laboratory to discover theorems of geometry.

Our Geometry in Egypt and China. William A. Austin. Mathematics Teacher, Vol. 16, pp. 78–86, February, 1923.

Laboratory method of teaching plane geometry with the use of manuals. No uniform plan of recitation. Pupils are actively at work and take the initiative.

Geometry a Laboratory Science. William A. Austin. School Science and Mathematics, Vol. 24, pp. 58–71, January, 1924.

"Two large rooms are furnished for the laboratory work. Each room has tables and chairs. Regular recitation rooms containing ordinary desks are also used for this purpose. In each room there is a cabinet used for filing away the work of students. . . .

"Each pupil has a drawing board, a T-square, a straightedge and scale, a protractor, a compass, a triangle, and a copy of the manual.

"Two days per week the pupils have a double (laboratory) period. Pupils follow the directions in the manual and perform geometric experiments and write up the notes on what are called plates. Single periods three days per week are used for recitation work." Illustrations of work.

A Laboratory Method of Teaching Mathematics in the Classroom. Charles A. Stone. Mathematics Teacher, Vol. 17, pp. 209-222, April, 1924.

Observation, experimentation, and manipulation are favored rather than just the textbook medium. Problems in this method are described.

Geometry a Laboratory Science. Joseph A. Nyberg. School Science and Mathematics, Vol. 24, pp. 948-953, December, 1924.

A criticism of William A. Austin's paper in School Science and Mathematics, Vol. 24, pp. 58-71, January, 1924.

Geometry a Laboratory Science. F. H. Sutton. School Science and Mathematics, Vol. 24, pp. 954-956, December, 1924.

Discusses William A. Austin's paper in School Science and Mathematics, Vol. 24, pp. 58-71, January, 1924.

Geometry, a Laboratory Science. John O. Pyle. School Science and Mathematics, Vol. 24, pp. 956-957, December, 1924.

A discussion of William A. Austin's paper in School Science and Mathematics, Vol. 24, pp. 58-71, January, 1924.

Teaching Plane Geometry without a Textbook. Theodore Strong. Mathematics Teacher, Vol. 19, pp. 115-119, February, 1926.

The Laboratory Method in Teaching of Geometry. C. A. Austin. Mathematics Teacher, Vol. 20, pp. 274-279, May, 1927.

Discusses attack, procedure, and teaching technique in the laboratory plan.

Laboratory Work in Geometry. R. M. McDill. Mathematics Teacher, Vol. 24, pp. 14-21, January, 1931.

"In laboratory geometry, the pupil works (for the time being, at least) as an individual. He works with his hands, at a table, using square, protractor, compass, rule, and scissors. An experiment is a question put to nature. The scientific spirit demands that we be on the alert to put questions to nature and that we be honest in recording the results we really get, not necessarily what we think we ought to get.

The Mathematics Laboratory—a Device for Vitalizing Mathematics. J. A. Ramseyer. Mathematics Teacher, Vol. 28, pp. 228-233, April, 1935.

The room—its equipment, bulletin board, books, instruments, and apparatus.

Progress in the Correlation of Physics and Mathematics. F. L. Bishop. School Science and Mathematics, Vol. 5, pp. 152-159, March, 1905.

"Another question which is often asked and which is certainly much to the point is: Do these students know their pure mathematics as well as students who have had only the abstract mathematics? This is, of course, a very difficult question to answer. I have made an attempt to find an answer in this way.

"I have in my classes not only students who have taken this work, but also students who have come to us from first-class high schools where I know that the preparation in pure mathematics is very good. I have sometimes asked for the proof of some geometrical theorem that we have been using, as for instance the Pythagorean theorem. I have never yet found a student with only the abstract preparation who would attempt to demonstrate one of these offhand. While I have found that a large number of the students who have had the concrete geometry were able to give the demonstration . . . as Dr. Milliken very aptly expresses it, 'you can teach all the physics you want in algebra and geometry and then there will be plenty left for us'."

An Attempt to Correlate Algebra and Physics. Willard S. Bass. School Science and Mathematics, Vol. 6, pp. 495–500, June, 1906.

Gives a proposed outline of course in physics and algebra.

A General Science Course of Elementary Physics and Mathematics Combined. J. C. Gray. School Science and Mathematics, Vol. 12, p. 377, May, 1912.

Work in physics and mathematics combined in experiments and problems.

THE MATHEMATICS CLASSROOM AND ITS EQUIPMENT

A Mathematical Atmosphere. W. D. Reeve. Mathematics Teacher, Vol. 31, p. 387, December, 1938.

Editorial discusses pictures and equipment for every mathematics classroom.

Planning the Mathematics Classroom. Fred L. Bedford. The School Executive, pp. 290–292, April, 1936.

List of suggested equipment includes chart files, shelves and table drawers, bulletin board, blackboard tools, teacher's file, museum case and show case, models, instruments, and other equipment. Architects' sketches for a mathematics laboratory are included.

A Mathematics Room That Speaks for Itself. Edith L. Mossman. School Science and Mathematics, Vol. 33, pp. 423–430, April, 1933.

Wall charts made by students include one listing seven requirements for "An Excellent Paper," nine "Desirable Habits to Be Formed or Strengthened in a Mathematics Course," another on "Earliest Mathematics, ?–1700 B.C." List of 24 other student-made posters is included.

Stimulating Interest in Mathematics by Creating a Mathematical Atmosphere. Mary Ruth Cook. Mathematics Teacher, Vol. 24, pp. 248–254, April, 1931.

This article gives suggestions on how a mathematical atmosphere can be created. Some means are the use of mathematical models, pictures, proverbs, reading room, posters, exhibits.

The Bulletin Board as a Teaching Aid. G. P. Cahoon. School Science and Mathematics, Vol. 28, pp. 867–873, November, 1928.

Checking and clipping articles, filing clippings, the filing boxes, folders, background for mounting, attaching clippings to background, titles, a "wire" bulletin board, portable bulletin boards, time between changes, using the bulletin board, type of questions, periodicals and sources of material, outcomes.

Fugitive Materials. Frances A. Mullen. Mathematics Teacher, Vol. 31, pp. 205–208, May, 1938.

Establishing a vertical file with major headings: biographies, other historical notes, puzzles, other recreations, pictures, plays, club programs, humor, importance—general, relation to other fields, miscellaneous, to be filed. Sources given. Students contribute and maintain file.

Mathematical Equipment and Its Uses. H. C. Wright. School Science and Mathematics, Vol. 15, pp. 500–504, June, 1915.

At University of Chicago High School each mathematics room contained two large wooden protractors, two wooden 30°–60°-right triangles, two isosceles right triangles, five wooden straightedges, nine blackboard compasses, fourteen foot-rulers, ten lead pencils, two small brass protractors, a spherical blackboard, a five-foot slide rule, a cork bulletin board. Glass doored cases for storing of mechanical devices and models. A case containing twenty-five revolving panels, a section of squared blackboard, and a supply of crayons; scissors and string also available. Surveying equipment included two transits, level, leveling rod, steel chain, several steel tapes, wire pins, red and white sight rods, and engineer's notebooks for field work.

The Primary Purpose of Training in Mathematics—Not a Kit of Tools but a Way of Thinking. Edith L. Mossman. School Science and Mathematics, Vol. 38, pp. 992–1002, December, 1938.

Discusses well equipped classroom: plenty of blackboard space, a library, pictures of mathematicians, slide rules, transit, models, charts and posters, etc.

A Method of Making Wall Charts. N. A. Harvey. School Science and Mathematics, Vol. 14, p. 516, June, 1914.

A negative is made of picture or chart desired, and enlarged to a convenient size—e.g., 20" x 26"; this is attached to a mounting board 22" x 28".

See also BOOKS:

Enriched Teaching of Mathematics. M. Woodring and V. Sanford. Pp. 104–112.

MATHEMATICS CLUBS

A High School Mathematics Club. Charles W. Newhall. School Science and Mathematics, Vol. 5, pp. 323–330, May, 1905.

Describes activities of one of the first secondary school clubs in the United States, that at Shattuck School, Faribault, Minn. Contains outline of several topics and suggestions for meetings.

Sources of Program Material and Some Types of Program Work Which Might Be Undertaken by High School Mathematics Clubs. Ruth Hoag. Mathematics Teacher, Vol. 24, pp. 492–502, December, 1931.

Projects include compilation of dictionary of terms and symbols used in high school mathematics, mathematics scrapbook, construction of instruments for field work, writing of a history of mathematics for high school students, construction of a set of models for solid geometry classes, preparation of mathematics paper. Demonstrations and experiments include soap bubbles and mathematics, finding of pi by chance, paper folding and plane geometry, slide rule demonstration, the steel square. Also includes a list of plays, skits, and dialogues, mathematical games, topics for group or individual presentation.

Mathematics Club Programs. A. Harry Wheeler. Mathematics Teacher, Vol. 16, pp. 385–390, November, 1923.

Lists fifteen topics used in club programs. Also gives sixty-eight titles and references for topics in number and measuring systems, the metric system, history of arithmetic, old arithmetics, some remarkable numbers, curious properties and beliefs, and geometry.

Making Mathematics Interesting. Augusta Barnes. Mathematics Teacher, Vol. 17, pp. 404–410, November, 1924.

Informal club meetings and contests as part of class work. Bibliography.

Junior High School Mathematics Clubs. Ruth Person. Mathematics Teacher, Vol. 34, pp. 228–229, May, 1941.

Mathematical recreations and games are used as basis for club meetings. References.

A Mathematics Club. Mary Caroline Hatton. Mathematics Teacher, Vol. 20, pp. 39–45, January, 1927.

Activities include a slide rule project, a surveying study, celebration of Mathematics Week, transit operation, and films relating to mathematics and its applications.

High School Mathematics Clubs. Zulu Reed. Mathematics Teacher, Vol. 18, pp. 341–362, October, 1925.

Discusses a program on the slide rule, including instructions for constructing a rule; tricks and puzzle problems such as tricks with cards and numbers; eight number rhymes; songs, contests, and games including puns from geometry; a short play, "The Math Quest"; a hike into the woods to study mathematics in nature; a debate on the metric system. Annotated bibliography.

Mathematics Clubs. Helen Russell, Minerva Duncan, Rebecca Symes, and Mary Derby. Mathematics Teacher, Vol. 17, pp. 283–285, May, 1924; pp. 350–358, October, 1924.

How to organize a club; such committees as entertainment, library, history, vocational, problem, "student help," contest, prize. Several programs are outlined. Bibliography.

Mathematics Clubs in the High School. Sophia Refior. Mathematics Teacher, Vol. 15, pp. 434–435, November, 1922.

Discusses a few typical programs.

Mathematics Clubs. Louisa M. Webster. Mathematics Teacher, Vol. 10, pp. 203–208, June, 1917.

Activities of Hunter College club are reported. Topics include anagyphs, Lincoln's debt to mathematics, the watch as a compass, alligation, use of imagination in mathematics, mathematicians famous in other fields, the meaning of a billion, mathematics in nature.

Mathematics Clubs in the High School. C. A. Snell. Mathematics Teacher, Vol. 8, pp. 73–78, December, 1915.

Mathematics Clubs. Frank C. Gegenheimer. School Science and Mathematics, Vol. 16, pp. 791–792, December, 1916.

Weekly meetings at which members proposed problems for solution and read articles in magazines relating to mathematics. Answers the question, "What shall we do for the bright pupil?"

Mathematics Clubs. H. Vernon Price. Mathematics Teacher, Vol. 32, p. 324, November, 1939.

Activities suitable for a club with limited membership based on honor grades.

Mathematics Clubs in Secondary Schools. Beulah Shoesmith. School Science and Mathematics, Vol. 16, pp. 106–113, February, 1916.

Describes programs and topics discussed at meetings.

The Enrichment of the Mathematics Course. H. E. Riter. Mathematics Teacher, Vol. 31, pp. 3–6, January, 1938.

Activities which can be carried on as class work or in the mathematics club: surveying, maps and scale drawings, etc. References.

The Mathematics Club at Curtis High School. Esther Sweedler. Mathematics Teacher, Vol. 29, pp. 394–395, December, 1936; also in High Points, Vol. 19, pp. 62–64, February, 1937.

Studying surveying as an activity of the mathematics club.

The Mathematics Club—Streamlined. Max F. Weiss. High Points, Vol. 21, pp. 74–77, October, 1939.

Students sponsored a Math Information Please program, math scavenger hunt, a program on logical analysis, mathematical guggenheim (a parlor game called 20 questions of "guggenheim"), math bingo, topolunacies, mathematical charades, mathematical Professor Quiz.

High School Mathematics Club. Radio talk by Norman Anning. Mathematics Teacher, Vol. 26, pp. 70–76, February, 1933.

Hints about organization: Start a club library. Put on an assembly program. Try problems in problem departments of mathematics magazines. Models for the school display cabinet.

Monroe Surveyors' Club. Morris Kaplan. High Points, Vol. 22, pp. 55–56, June, 1940.

Recreational Values Achieved through Mathematics Clubs in Secondary Schools. Marie Gugle. Mathematics Teacher, Vol. 19, pp. 214–218, April, 1926.

Recreations for the Mathematics Club. Bryon Bentley. Mathematics Teacher, Vol. 23, pp. 95–103, February, 1930.

Discusses several which can be used as stimulus to class discussion or at club meetings. Thirty recreations chosen from algebra, geometry, and trigonometry.

A Christmas Project. Muriel Batz. Mathematics Teacher, Vol. 30, pp. 376–377, December, 1937.

A Christmas tree was decorated with geometrical figures and solids as ornaments. Suggests a tree which is entirely geometrical: the base a cylinder, the stem a very narrow rectangular parallelepiped, the main part of the tree consisting of several tiers of graduated trapezoids surmounted at the top by an isosceles triangle.

The Christmas Party. Phyllis A. Holroyd. Mathematics Teacher, Vol. 32, pp. 352–353, December, 1939.

Includes picture of a mathematical Christmas tree decorated with various star-polyhedrons.

A Mathematical Christmas Tree—a Picture. The Mathematics Teacher, Vol. 33, p. 337, December, 1940.

The Math Star. Frances A. Mullen. Chicago Schools Journal, Vol. 19, pp. 169–172, March-April, 1938.

Describes math magazine published by students, methods of organization, types of articles and problems, and other topics.

See also BOOKS:

Mathematical Clubs and Recreations. S. I. Jones.

Enriched Teaching of Mathematics. M. Woodring and V. Sanford. Pp. 85–100.

THE MATHEMATICS LIBRARY

The Mathematics Library and Recreational Programs. Helen Taylor. School Science and Mathematics, Vol. 30, pp. 626–634, June, 1930.

Contains list of thirty titles of books with short description of each, a description of three high school assembly programs, including an assembly sketch and song.

A List of Reference Books and Magazines for Teachers of Mathematics. W. D. Reeve. Mathematics Teacher, Vol. 15, pp. 303–307, September, 1922.

Reference books and magazines listed by topics: books of a pedagogical nature;

books on teaching of mathematics; books relating to mathematical topics; book containing recreational material; books of a historical nature.

Bibliography of Popular Mathematics. D. B. Lloyd. School Science and Mathematics, Vol. 38, pp. 186–193, February, 1938.

Books on history, biography, recreations, enrichment, and source material. Periodicals on appreciation of mathematics, relation of mathematics to other fields, history, mathematicians, arithmetic, numbers, algebra, geometry, trigonometry, probability, measurement, physics and astronomy and the fourth dimension, recreations and problems, devices and equipment, programs and contests, plays.

New Materials and Equipment in the Teaching of Mathematics. B. R. Ullsvik. School Science and Mathematics, Vol. 39, pp. 432–442, May, 1939.

Discusses and lists books and magazine articles, motion pictures, posters, puzzles, equipment, evaluation instruments, records, and commission reports.

A Selected List of Mathematics Books for Colleges. Elton J. Moulton. American Mathematical Monthly, Vol. 48, pp. 600–609, November, 1941.

FILMS, SLIDES, AND PROJECTION EQUIPMENT

FILMS

The Play of Imagination in Geometry. Film produced by David Eugene Smith and Aaron Bakst. Review by W. S. Schlauch. Mathematics Teacher, Vol. 24, pp. 55–56, January, 1931.

Mathematics Films. Margaret Punnett. The Mathematical Gazette, Vol. 21, pp. 149–151, May, 1937.

Exhibition of Mathematical Films. The Mathematical Gazette, Vol. 20, pp. 110–114, May, 1936.

Describes mathematical films exhibited at a meeting in England in 1936.

Mathecinematics. Grace M. Hopper. American Mathematical Monthly, Vol. 47, pp. 565–568, October, 1940.

Animated cartoon technique is used to produce film showing limacons.

Mathematics Films. E. H. C. Hildebrandt. Mathematics Teacher, Vol. 34, pp. 26–34, January, 1941.

Films are listed under headings: geometry, advanced mathematics, teaching of mathematics, mathematics and art, mathematics and architecture, mathematics and nature, weights and measures, money, mathematics and physics, optical instruments, engineering and industry, mathematics and astronomy.

A Science Teacher Looks at the Classroom Film. H. Emmett Brown. School Science and Mathematics, Vol. 39, pp. 342–351, April, 1939.

Criticisms: that educators tend to claim too much from the use of the motion pictures in the classroom and embrace the new in visual education too eagerly. Too many films fail to recognize that the unique function of film is to show motion; they cover too much ground, are too inclusive, too general, and often downright dull; they are too inaccurate and do not balance the amount of detail. There are often small but irritating technical errors or omissions and the photography is often poor. Animated drawings are not employed to best advantage. Advertising in some films is objectionable; titles of films are often misleading and the editing poor. Projectors are too expensive and too cumbersome.

The Classroom Film. R. E. Davis. School Science and Mathematics, Vol. 39, pp. 627–630, October, 1939.

Replies by a chemistry teacher to criticisms of H. Emmett Brown in School Science and Mathematics, April, 1939. Impossible to produce a picture that will please every teacher. Film cannot take the place of a teacher. Film can show certain types of experiments which the busy teacher cannot prepare. Visits to industrial plants are sometimes impossible because of large classes. The classroom film has its limitations. The use of films will not lessen the teacher's load.

SLIDES

Lantern Slides. Fred L. Holtz. School Science and Mathematics, Vol. 6, pp. 262–267, April, 1906.

A Small Size Lantern Slide. Clarence R. Smith. School Science and Mathematics, Vol. 29, pp. 530–532, June, 1929.

Suggestions for Making 2 x 2 Inch Lantern Slides with Inexpensive Equipment. Victor E. Schmidt. School Science and Mathematics, Vol. 40, pp. 165–169, February, 1940.

Lists equipment and materials needed, discusses method of making slides and construction of the camera and holder.

Homemade Lantern Slides in the Teaching of Plane Geometry. N. L. Martin. Educational Screen, Vol. 13, January, 1934.

Sixty slides on congruency of triangles, parallel lines, and other straight-line figures were made, together with sixty on circles and loci. Some slides were cellophane and others were etched on glass. Slides were used to introduce a new unit, as well as to show its development and to carry on the review. Author's conclusion from study: "1. A series of correlated lantern slides will increase the effectiveness of the presentation of plane geometry. 2. The value of lantern slides appears to vary inversely with the native capacity of the individual child. 3. There is a high degree of correlation between achievement and the use of the slides."

The Stereopticon as an Aid to Physics Teaching. Charles F. Valentine. School Science and Mathematics, Vol. 28, pp. 78–80, January, 1928.

"Only two or three slides are selected for any class discussion period. The slides must be selected with particular attention to their use in the lesson plan. The slides must have a short focus and throw a brilliant image on a blackboard. The stereopticon must have a short focus and throw a brilliant image on a blackboard; it must be satisfactory under ordinary lighting conditions, and should be located on the instructor's desk and be instantly available at all times by merely pressing a button." The blackboard is used as a screen; the room is not darkened. "The question is not how many slides in a class hour, but how they are presented."

STEREOGRAMS

Geometric Stereograms—A Device for Making Solid Geometry Tangible to the Average Student. Walter Shenton. Mathematics Teacher, Vol. 8, pp. 124–131, March, 1916.

Directions given for producing figures of geometric solids with red and blue colored pencils. These are viewed with red and blue stereoscopic spectacles. Can be produced by students.

Stereoscopic Harmonic Curves. William F. Riggs. School Science and Mathematics, Vol. 24, pp. 29–36, January, 1924.

Scientific Use for the Stereoscope. Aid to the Imagination in Picturing Space Models. Scientific American Supplement, Vol. 72, p. 117, August, 1911.

See also BOOKS:

Excursion in Mathematics, E. R. Breslich.

Basic Principles of Analytic Geometry. W. A. Spencer.

Diagrams in Three Dimensions for Solid Geometry. E. R. Breslich.

Mathematical Snapshots. H. Steinhaus. Pp. 42, 44, 68, 72, 73, 77, 79, 84, 85, 89–92, 94, 100, 102, 105.

Les Anaglyphes Geometriques. H. Vuibert.

EQUIPMENT

A Universal Projection Lantern. E. J. Rendtorff. School Science and Mathematics, Vol. 9, pp. 293–298, March, 1909.

Directions for constructing a lantern which projects photographic slides, microscopic slides, apparatus both transparent and opaque by either horizontal or vertical projection, and printed diagrams.

A Reader for Microfilm. Herman Branson. School Science and Mathematics, Vol. 40, pp. 411–412, April, 1940.

A visual magnifier made from a commercial binocular viewer for 35 mm. film strips at a total cost of about eighty cents. Photograph shows viewer in use.

Microfilm Equipment for the Individual Worker. Herman Branson. School Science and Mathematics, Vol. 41, pp. 140–143, February, 1941.

A microfilm camera apparatus is assembled from a 35 mm. camera and an enlarger stand. Focusing, choice of speeds, and film stock needed are discussed. Two illustrations show the completed microfilm camera and the projector ready for use. Picturols are being made at a cost of about 6¢ a frame.

A Pupil Operator Service for the Projection of Visual Aids. Walter W. Bennett and Lewis S. Edgarton. School Science and Mathematics, Vol. 36, pp. 356–363, April, 1936.

Description of a visual aid system of a Rochester, New York, high school which makes use of student service in operating the projection machine and keeping it in repair, and employs students for the clerical work of assigning the machine to teachers and procuring films requested. A chart shows relationships of the Visual Aids Corps to other groups whereby it becomes a correlated extra-curricular activity.

See also BOOKS:

Enriched Teaching of Mathematics. M. Woodring and V. Sanford. Pp. 89–95.

CONTESTS

A Mathematics Contest—Its Relation to the General Problem of Individual Differences. Raleigh Schorling. School Science and Mathematics, Vol. 15, pp. 794–797, December, 1915.

Rules agreed upon for a contest between two high school teams of six students each chosen from first-year classes. A written and an oral contest were held.

Mathematics Relays for High Schools. Ernest H. Koch, Jr. and Thomas H. McCormick. Mathematics Teacher, Vol. 8, pp. 116–123, March, 1916; also in School Science and Mathematics, Vol. 16, pp. 530–536, June, 1916.

Class divided into teams of four members each. All teams work simultaneously. Practice is given in the four arithmetic processes. In addition, for example, the teacher will dictate two five-digit numbers. The first member finds their sum, the second member adds the sum and the second number, the third member adds the first and second sums, the fourth member adds the second and third sum, the first member adds the third and fourth sum, etc., for ten cases or so. Similar problems are worked in subtraction, multiplication, and division. All-school teams are chosen and these may compete in interschool contests.

A Mathematical Contest. Edgar T. Boughn. School Science and Mathematics, Vol. 17, pp. 329–330, April, 1917.

Rules and type problems used in a contest between Manual Arts High School in Los Angeles and the Pasadena High School.

A New Form of School Contest. J. T. Rorer. Educational Review, Vol. 57, pp. 339–345, April, 1919.

A track meet in mathematics.

Mathematics Teams to Motivate Drill Work in Junior High Schools. R. V. Kessler. Mathematics Teacher, Vol. 27, pp. 25–29, January, 1934.

Includes a list of contest rules for a tournament and suggestions for conducting competitions.

A Computing Club. School Science and Mathematics. Vol. 31, p. 1037, December, 1931.

Editorial suggests organizing a rapid calculating club in schools. "It may prevent a failure in college mathematics or science."

Mathematics Contests. Ernest H. Koch, Jr. Mathematics Teacher, Vol. 9, pp. 179-186, June, 1917.

Discusses nature of contests and describes a relay competition in algebra and a "crisscross" relay in addition.

A Geometric Recreation. Isabel Harris. School Science and Mathematics, Vol. 20, pp. 731–733, November, 1920.

Rules set up for six contests in a class in geometry.

A Mathematical Contest. Helen M. Walker. Mathematics Teacher, Vol. 20, pp. 274–279, May, 1927.

At a meeting of mathematics teachers, forty-two articles were placed on exhibit and those present were required to identify a term from elementary mathematics. Can be used by a high school or college mathematics club, for a school exhibit or a meeting of teachers.

An Algebra Baseball Game. Mathematics Teacher, Vol. 23, pp. 317–320, May, 1930.

Rules are given for a "classroom" game.

See also BOOKS:

Enriched Teaching of Mathematics. M. Woodring and V. Sanford. P. 86.

EXHIBITS

Exhibit of High School Mathematics—Its History and Educational Value. W. D. Reeve. School and Society, Vol. 2, pp. 191–197, August 7, 1915.

A Mathematical Atmosphere. Olive A. Kee. The Third Yearbook of the National Council of Teachers of Mathematics, 1928. Pp. 268–276.

The Mathematical Collection. Georg Wolff. The Eighth Yearbook of the National Council of Teachers of Mathematics, 1933. Pp. 216–243.

Models illustrating the parabola and paraboloids, applications in geometry and algebra. Fifty-three figures.

Helping Mathematics with an Exhibit. William C. Krathwohl. Mathematics Teacher, Vol. 31, pp. 68–69, February, 1938.

Exhibits sponsored by Chicago high schools at Adler Planetarium have as their object "to show the eternal and everlasting utilitarian character of mathematics both from the pragmatic as well as the aesthetic points of view. Exhibit articles are made by students; the exhibit is changed every quarter and is built around a central theme. Publicity given through daily papers and other mediums. List of themes such as applications of high school mathematics to games of chance, multiplication without thinking, using mathematics to deceive, how mathematics is used in churches, how the designer of dresses uses mathematics.

Using and Preparing a Mathematics Exhibit. William C. Krathwohl. School Science and Mathematics, Vol 39, pp. 702-706, November, 1939.

Discusses mathematics exhibits prepared by Chicago high schools and held at the Adler Planetarium. Model bridges were made by students showing how mathematics was used in their construction. Interest in making models can be aroused through spirit of competition. Reading material in exhibit should be in letters large enough to be read ten feet away by person wearing bifocal glasses. Models should be plainly marked and, as far as possible, in words of one syllable. Generous use should be made of colors. "Gorgeous reds, oranges, blues and greens, if they do not clash, radiate happiness and cheerfulness. . . . The keynotes of a successful exhibit are simplicity, ease of observation and attractiveness."

Mathematics Exhibit at Adler Planetarium. H. McCartney. Junior College Journal, Vol. 9, pp. 251-253, February, 1939.

Mathematics Exhibit—World's Fair, Chicago. Lillian Moore. Mathematics Teacher, Vol. 26, pp. 482–486, December, 1933.

Lists exhibits in arithmetic and algebra; applied mathematics; analysis; geometry; Johansson's gauges; mathematical recreations, including the Mystic Number Machine which obtained an unknown number through the application of magic squares; Pollock's Models—string figures on frames; skew curve projection; centripetal force; applied centripetal force; Galton quincunx; computation of pi; ellipsograph; mathematical machines, such as the harmonic motion apparatus, Michelson's harmonic analyzer and synthesizer, Gonella's planimeter, Lehmer's machine illustrating the factorization of a number; gyro-compass repeater; aerodynamics, pantographer.

Mathematics Exhibit at the Century of Progress Exposition. B. Braverman. High Points, Vol. 16, pp. 30–31, December, 1934.

Visual Aids at the World's Fair—A Royal Road to Knowledge. C. Fordney. Nation's Schools, Vol. [illegible], pp. 55–57, May, 1934.

Museum of Science and Industry, Jackson Park, Chicago, Illinois.

Has a mathematics exhibit consisting of Galton's quincunx, a device for evaluating pi, ellipsograph, a slide rule exhibit, a Michelson harmonic analyzer of twenty terms, an exhibit using a photronic cell for the summation of a series whose terms are represented by equally spaced lights, a number of models of ruled surfaces, a rotating cone in a plane beam of light to show the conic sections, a device for evaluating pi by tossing a coin on a checkerboard, and a pair of Blackburn pendulums for the production of Lissajous' figures with sand.

Dr. David Eugene Smith's Gift of Mathematical Instruments to Columbia University. School and Society, Vol. 43, pp. 313–314, March, 1936.

Poster of the Mathematics Exhibit at the Horace Mann School for Girls. Mathematics Teacher, Vol. 23, p. 35, January 1930.

A Mathematics Exhibit. M. Cottell Gregory. Mathematics Teacher, Vol. 23, pp. 382–384, October 1930.

Graphs involving local statistics; designs for linoleum, wallpaper, doors, and windows; polyhedrons; a solid geometry church with hexagonal pyramid tower and triangular prism roof; geometry in the houses of primitive people.

A Unique Mathematics Exhibit. Ruth Wilson. Mathematics Teacher, Vol. 30, pp. 128–129, March, 1937.

Under the title "The Practical Applications of Mathematics in Various Occupations and Industries," the mathematics department held an exhibit based on local uses. The electric and power company cooperated in making several charts, as did the telephone company, life insurance company, city engineering department, railroad, and local architect.

Homemade Mathematics Exhibit. William Sell. American Mathematical Monthly, Vol. 40, pp. 555–556, November, 1933.

Exhibit Your Mathematics. L. M. Morris. High Points, Vol. 16, pp. 22–25, December, 1934.

Mathematics Display at a School Exhibition. School, Vol. 28, pp. 136–137, October, 1939.

The Exhibit as a Supplementary Method. Crystallography. Harold J. Abrahams. School Science and Mathematics, Vol. 36, pp. 950–956, December, 1936.

Exhibit in which chemical and mineral crystals are grouped, according to formation, with celluloid models. Description of the six main groups into which all crystals may be divided. Directions for making the models and preparing the crystals.

Project—Techniques Relative to Exhibitions. Maitland P. Simmons. School Science and Mathematics, Vol. 40, pp. 434–438, May, 1940.

Written for a science exhibit, there are many suggestions which can be used in the mathematics exhibit. Sufficient time must be allowed. Creation of desire for this work is essential. Projects should be workable, unusual, and varied. Standard size recommended is two feet square and not over three feet high. Definite check-ups of progress must be made. Projects should be submitted weeks before the exhibit. Explanatory posters should serve as guide.

Mathematical Instruments. David Baxandall. Encyclopædia Britannica, 14th Edition, 1939, Vol. 15, pp. 69–72.

Mathematical Models. G. W. Cussons. Encyclopædia Britannica, 14th Edition, 1939, Vol. 15, pp. 72–75.

See also BOOKS:

Enriched Teaching of Mathematics. M. Woodring and V. Sanford. Pp. 50, 101–103, 108, 110–111.

PLAYS

Archimedes, a Mathematical Genius. Noma Pearl Reid. School Science and Mathematics, Vol. 41, pp. 211–219, March, 1941.

A play developed as a creative activity in a mathematics class.

The Eternal Triangle. A playlet. Gerald Raftery. Mathematics Teacher, Vol. 26, pp. 85–92, February, 1933.

Everyman's Visit to the Land cf the Mathematicians. Edith Bruce Paterson. Mathematics Teacher, Vol. 31, pp. 7–18, January, 1938.

"An imaginary journey to the abode of some of the great mathematicians." Includes production notes regarding scenery, posters, properties, and costumes. The episodes are from geometry, applications of mathematics in ancient times, arithmetic, algebra, mathematics in the Middle Ages, and the foundations of modern mathematics. Bibliography.

Falling in Love with Plain Geometry. A comedy in two acts. Caroline Hatton and Doris H. Smith. Mathematics Teacher, Vol. 20, pp. 389–402, November, 1927.

Includes list of properties and directions for dances and musical numbers.

Flatland—a Mathematics Play. School Science and Mathematics, Vol. 14, pp. 583–587, October 1914.

A description of a play reported in the Mathematical Gazette. It is based on the book, "Flatland." Characters are listed and descriptions given for five scenes. The audience views the play from the balcony of the auditorium.

Geometry Humanized. A school play in one act. Erma Scott. Mathematics Teacher, Vol. 21, pp. 92-101, February, 1928.

It Can't Happen Here. A mathematical musical farce. A. Marie Whelan. American Mathematical Monthly, Vol. 45, pp. 617-628, November, 1938.

A Little Journey to the Land of Mathematics. Alma E. Crawford. Mathematics Teacher, Vol. 17, pp. 336–342, October, 1924.

A play in three acts based on "King Math Rules the World."

A Mathematical Fantasy. Elizabeth B. Cowley. School Science and Mathematics, Vol. 33, pp. 535–536, May, 1933.

A Mathematical Victory. A play in two acts. School Science and Mathematics, Vol. 17, pp. 475–482, June, 1917.

Mr. Chips Teaches Geometry. John J. Shea. School Science and Mathematics, Vol. 40, pp. 720-726, October, 1940.

A play based on a locus problem.

Mock Trial of B versus A, or Solving a Personal Equation by the Judicial Process. Kathryn McSorley. School Science and Mathematics, Vol. 18, pp. 611-621, October, 1918.

Adopted from Stephen Leacock's story, "A, B, C."

A Number Play in Three Acts. T. Schlierholz. Mathematics Teacher, Vol. 17, pp 154–169, March, 1924.

Based on "Number Stories of Long Ago" by David Eugene Smith.

Out of the Past. Florence Brooks Miller. Mathematics Teacher, Vol. 30, pp. 366–370, December, 1937.

A play, in two acts, created and presented in a school assembly by a ninth grade class. Based on development of our number system and the use of mechanical devices for computation.

The Radical Dream. A playlet for puppets. J. M. Leps. School Science and Mathematics, Vol. 33, pp. 279-287, March, 1933.

X on the Spot. Louise Meals. Mathematics Teacher, Vol. 32, pp. 296-300, November, 1939.

When Wishes Come True. A mathematics play in two scenes. Hannah A. Parkyn. Mathematics Teacher, Vol. 32, pp. 16-24, January, 1939.

Professor Whiz and His Class in Magic Math. Idah Vaughn. School Science and Mathematics, Vol. 39, pp. 540–545, June, 1939.

For a junior high school assembly program. Nine or more characters for a quiz program.

See also BOOKS:

Enriched Teaching of Mathematics in the Junior and Senior High School. M. Woodring and V. Sanford. Pp. 87–89.

RECREATIONS

Recreational Aspects of Mathematics in the Junior High School. Anna R. Meeks. Mathematics Teacher, Vol. 29, pp. 202, January, 1936.

The use of mathematical puzzles and games in the class period, mathematics club, the mathematics newspaper.

Mathematical Recreations. Martha Pierce. Mathematics Teacher, Vol. 19, pp. 13–24, January, 1926.

Recreations in Secondary Mathematics. Charles W. Newhall. School Science and Mathematics, Vol. 15, pp. 277–293, April, 1915.

Methods of using recreations in mathematics clubs; in informal meetings devoted to games, puzzles, tricks; in assigned readings; in debates; in the classroom. References include over fifty books and sixty magazine articles.

Misplaced Mathematical Recreations. M. C. Bergen. School Science and Mathematics, Vol. 39, pp. 766–768, November, 1939.

Conclusion: "Practice in puzzle solving makes for proficiency in puzzle solving, but in nothing else."

From Interest to Interest. M. Weiner. Mathematics Teacher, Vol. 30, pp. 23–26, January, 1937.

Mathematical recreations books awakened interest in a boy and he became one of the outstanding students in the school.

Arithmetic Puzzles for Junior Grades. E. W. Montgomery. School (Elementary Edition), Vol. 29, pp. 809, May, 1941.

Proving a Geometrical Fallacy by Trigonometry. William W. Johnson. School Science and Mathematics, Vol. 19, pp. 527–528, June, 1919.

The famous dissection puzzle that $64 = 65$ i.e., a square 8 units on a side may be cut into four pieces which will form an apparent rectangle 5 x 13. The difference may be shown to be a parallelogram of area one square unit.

Mathematical Fallacies. Cecil B. Read. School Science and Mathematics, Vol. 33, pp. 586–589, June, 1933.

Discusses several common algebraic and geometric fallacies.

More Mathematical Fallacies. Cecil B. Read. School Science and Mathematics, Vol. 33, pp. 977–983, December, 1933.

More interesting fallacies which can be used in mathematics classes. References.

See also BOOKS:

Enriched Teaching of Mathematics. M. Woodring and V. Sanford. Pp. 96–99.

GAMES

Host. Walter W. Carnahan. School Science and Mathematics, Vol. 28, pp. 604–608, June, 1928.

A game played on a checkerboard and using pyramids, cubes, hexagonal prisms, cylinders, spheres, triangles, squares, hexagons, and circles for figures. Moves of figures are based on geometric shape. Rules are included. Pieces can be made with a few tools and little expenditure of time.

I Doubt It—a Mathematical Card Game. Irene Price. American Mathematical Monthly, Vol. 49, p. 117, February, 1942.

Mathematical Games. Alfreda Raster. Mathematics Teacher, Vol. 17 pp. 422–425, November, 1924.

Describes a version of "Old Maid in Algebra," a blackboard relay, crossed words, zip-zip, "Simon says, *Buzz*," and the game of math shark.

A Mathematical Crossword Puzzle. H. C. Cozard. School Science and Mathematics, Vol. 26, pp. 316, 318, March, 1926.

Involves such mathematical terms as sum, limit, number, problem, theorem, degree, circles, log, factor, sphere, polygon. Solution included.

BOOKS

MODELS

Anfertigung Mathematischer Modelle. K. Giebel. Leipzig: B. G. Teubner, 1925. 52 pages.

Elementary surveying instruments, model for bisecting an angle, variation of parts of right triangle, measurement of angles related to circles, pantographs, similar triangle figures, binomial theorem cube, polyhedrons, slide rules. Also discussions on how to use wood, paper, cardboard, glass, celluloid, clay, lead, and brass.

Early Science in Oxford. Part II: Mathematics. R. T. Gunther. New York: Oxford University Press, 1922. 101 pages.

Descriptions of early scales and protractors, spherical blackboards, mathematical models, geometric solids, calculating apparatus, drawing instruments, proportional compasses, parallel rulers, parabolic compasses, elliptical trammels, sectors, slide rules, standard measures, gauging instruments, weights, quadrants, transversals, and micrometers. Well illustrated.

General Plastics. Raymond Cherry. Bloomington, Ill.: McKnight and McKnight, 1941. 128 pages.

Use of plastics in making models.

Katalog Mathematischer und Mathematisch-Physikalischer Modelle, Apparate und Instruments. Walther Dyck. Munich, 1892. Vol. I, 430 pages: Vol. II, 135 pages.

Slide rules, calculating machines, devices for drawing various curves, quincunx of Galton, instruments for harmonic analysis and advanced mathematics.

Mathematical Models. Arnold Emch. Urbana: University of Illinois Press. Four pamphlets.

String models of surfaces of higher mathematics.

Mathematical Snapshots. H. Steinhaus. New York: G. E. Stechert and Company, 1939. 135 pages.

Linkages, dissection figures, polyhedrons, perspective, paper folding, stereograms, soap films, string surfaces, Mobius surfaces, knots, quincunx, and graphs.

Plastics in the School and Home Workshop. A. J. Lockrey. New York: D. Van Nostrand and Company, 1940. 233 pages.

Puzzles in Wood. E. M. Wyatt. Milwaukee: Bruce Publishing Company, 1928. 64 pages.

Dissection figures, shuffle puzzles, Tower of Hanoi puzzle, cross and wedge puzzles.

Solid Geometry. L. Lines. London: Macmillan and Company, 1935. 292 pages.

Chapters on polyhedrons, semi-regular and star polyhedrons, and crystal forms.

The Fifty-nine Icosahedra. H. S. M. Coxeter, P. DuVal, H. T. Flather, and J. F. Petrie. Toronto: University of Toronto Press, 1938. 26 pages, 20 plates.

Vielecke und Vielflache. M. Bruckner. Leipzig: Teubner, 1900. 227 pages, 12 tables.

A classic on all forms of polyhedrons, including semi-regular and star polyhedrons. 146 illustrations and numerous diagrams.

TOOLS AND INSTRUMENTS

Adler Planetarium and Astronomical Museum. Philip Fox. Chicago: Adler Planetarium. 63 pages.

Describes the optical planetarium installed in Chicago. Also discusses and pictures various historical mechanical planetariums, astolabes, nocturnals, quadrants, octants and sextants, sundials, calendars, circumferenter, graphometer, mathematical and drawing instruments, and telescopes.

Ruler and Compasses. Hilda P. Hudson. London: Longmans, Green and Company, 1916. 143 pages.

Possible constructions, ruler constructions, ruler and compass constructions, standard methods, one fixed circle, and compasses only. Bibliography.

The Steel Square. How to Use Its Scales, How to Make Braces, Roof Construction, and Other Uses. G. Townsend. Chicago: American Technical Society, 1940. 96 pages.

Tools—A Mathematical Sketch and Model Book. Robert C. Yates. Baton Rouge: Louisiana State University Press, 1941. 194 pages, 79 plates.

Sections on the straightedge and modern compasses, dissection of plane polygons, paper folding, linkages, parallel and angle rulers, and other tools and linkages.

SLIDE RULE

A Manual of the Slide Rule, Its History, Principle, and Operation J. E. Thompson. New York: D. Van Nostrand Company, 1930. 232 pages.

Theory and operation of the rule, and modified forms of the Mannheim Rule—the Polyphase, Polyphase Duplex, Log-Log Duplex. Also history of the slide rule and such special forms as the circular and cylindrical models.

On the History of Gunter's Scale and the Slide Rule during the Seventeenth Century. Florian Cajori. Berkeley: University of California Press, 1920. Publications in Mathematics, Vol. I, No. 9, pp. 187-209.

A historical treatment of the slide rule.

Slide Rule Calculations. H. O. Cooper. New York: Oxford University Press, 1931. 132 pages.

Derivation of slide rule scales and theory of their use. Use made of slide rule in science and engineering is shown in chapter on applications. Appendix lists more than a dozen slide rules used in other fields.

The Slide Rule. E. R. Breslich and Charles A. Stone. Chicago: University of Chicago Press, 1929. 23 pages.

The Slide Rule. C. N. Pickworth. New York: D. Van Nostrand Company, 1941. 133 pages.

The Slide Rule and Logarithmic Tables. J. J. Clark. Chicago: Frederick J. Drake and Company, 1941. 219 pages.

SURVEYING

Field Work in Mathematics. C. N. Shuster and F. L. Bedford. New York: American Book Company, 1935. 168 pages.

Early instruments, scale drawing, approximate data, slide rule, angle mirror, hypsometer and clinometer, plane table, vernier, sextant, transit, pantograph.

Mathematics of Everyday Life, Geometry Unit. George A. Boyce and Willard W. Beatty. New York: Inor Publishing Company, 1936. 168 pages.

Experiments with elementary surveying devices at early high school level.

Surveying for Schools and Scouts. W. A. Richardson. London: George Philip and Son, Ltd., 1936. 110 pages.

An elementary approach to surveying through scale drawing and simple instruments.

STEREOGRAPHS

Basic Principles of Analytic Geometry. Walter A. Spencer. New York: Newson and Company, 1939. 27 pages, 12 plates, 22 figures.

Important solids of solid analytic geometry are shown.

Diagrams in Three Dimensions for Solid Geometry. E. R. Breslich. New York: Newson and Company, 1939. 15 plates.

Figures for theorems in solid geometry illustrated by means of stereographs.

Excursion in Mathematics. E. R. Breslich. New York: Newson and Company, 1938. 47 pages, 21 plates.

Using red and blue colored viewers, illustrations show cube, parallelepiped, prisms, cylinders, pyramids, cones, and spheres.

Les Anaglyphes Geometriques. H. Vuibert. Paris: Librairie Vuibert, 1912. 32 pages.

Fundamental principles of stereographs. Illustrations of dihedral angles, lines in space, volume of parallelepipeds, prisms, pyramids, star dodecahedron, cylinders, cone, spheres, and crystals.

Mathematische Raumbilder. O. Kohler, U. Graf, and C. Calov. Dresden: Ehlermann, 1938. 65 pages, 24 plates.

Volume of cube, binomial formula, regular solids, Cavalieri's principle, volume of pyramid as one-third of prism, conic sections, hyperboloids, the sine curve as the projection of the helix, spherical triangles, loxodromes.

Stereoscopic Photography—Its Application to Science, Industry and Education. Arthur W. Judge. London: Chapman and Hall, 1928. Available through American Photographic Publishing Company, Boston. 331 pages, 21 plates.

Interesting topics for amateur photographers: pseudo-stereo effects from single pictures by means of a homemade convex lens or convex mirror device, hand-drawn stereographs for solid geometry figures, stereographs produced from ordinary stereoscopic negatives, stereographs photographed with single lens or with stereoscopic cameras, stereoscopic aerial photography, projection of stereoscopic pictures. A chapter is devoted to stereoscopic aids to geometry teaching. Excellent bibliography.

GEOMETRICAL DRAWING

Geometrical Drawing and Design. J. Humphrey Spanton. London: Macmillan and Company, 1930. 262 pages.

Practical drawing based on theory of plane and solid geometry.

Geometrical Drawing for Art Students. I. H. Morris. London: Longmans, Green and Company, 1941. 228 pages.

Fundamental properties of designs and drawings.

Meyer's Handbook of Ornament—Geometrical and Floral. F. S. Meyer. Pelham, N. Y.: Bridgman, Publishers, 192[illegible]. 64 pages.

Source Book of Problems for Geometry. Mabel Sykes. Boston: Allyn and Bacon, 1912. 380 pages.

Tile and parquet floor designs. Gothic tracery based on forms in circles and on pointed forms, trusses, and arches. Diagrams, pictures, and exercises based on famous art and architectural designs of the world. Extensive bibliography.

Geometric Design. A. Bruce Ewer. Denver: Smith-Brooks Company, 1935. 28 pages.

TEACHING OF MATHEMATICS

Craftsmanship in the Teaching of Elementary Mathematics. F. W. Westaway. London: Blackie and Son, 1931. Available through Ryerson Press, Toronto. 665 pages.

Chapters on polyhedra, proportion and symmetry in art, the mathematical library and equipment, harmonic motion, map projection, orthographic projection; many teaching aids for the secondary school and junior college courses.

Mathematics for Junior High School Teachers. William L. Schaaf. Richmond: Johnson Publishing Company, 1931. 439 pages.

Problems in Teaching Secondary School Mathematics. E. R. Breslich. Chicago: University of Chicago Press, 1931. 348 pages.

The teaching of functions, formulas, and graphs; intuitive, demonstrative, and three-dimensional geometry. Illustrates various forms of teaching aids.

Teaching Mathematics in the Secondary Schools. J. H. Minnick. New York: Prentice-Hall, 1939. 336 pages.

Applications of mathematics and mathematics clubs are topics treated in two of the chapters.

The Case Against Arithmetic. E. M. Renwick. London: Simpkin Marshall, Ltd., 1935. 167 pages.

Chapters entitled "Misconceptions Relating to Measurement," "First Notions of Fractions," "The Fraction as Multiplier," "Difficulty in Finding Approximate Answers" discuss typical errors and the student comments experienced by every teacher of mathematics; also chapter on "Some Suggestions for Reform in the Teaching of Number."

The Teaching of Elementary Mathematics. Charles Godfrey and A. W. Siddons. Cambridge University Press, 1937. Available through Macmillan Company, New York. 322 pages.

Pythagorean Theorem by dissection. Three-dimensional work discussed under headings "Seeing a Solid Figure," "Drawing a Solid Figure."

The Teaching of Junior High School Mathematics. David Eugene Smith and William D. Reeve. Boston: Ginn and Company, 1927. 411 pages.

Homemade instruments discussed are those measuring distances and angles, those for drawings and constructions, including the T-square and carpenter's "square," and a circular blackboard. A device for illustrating the multiplication law of directed numbers is made from a light bar, small screw hooks, and weights. Chapters also on mathematics clubs and contests and mathematical recreations.

The Teaching of Mathematics. J. O. Hassler and R. R. Smith. New York: Macmillan Company, 1930. 405 pages.

The Teaching of Secondary Mathematics. Charles H. Butler and F. Lynwood Wren. New York: McGraw-Hill Book Company, 1941. 514 pages.

Materials of instruction—aids to teaching include textbooks and workbooks, equipment of the individual student and the classroom, instruments for field work, instruments for the classroom, such as models, spherical blackboard, drawing instruments, homemade equipment. Stimulating and maintaining interest in mathematics by motivation through intellectual curiosity, applications to other fields of study, mathematical clubs and recreations. Extensive bibliography.

The Technique of Teaching Secondary School Mathematics. Ernst R. Breslich. Chicago: University of Chicago Press, 1930. 239 pages.

Effective procedures and devices in teaching mathematics, arousing and maintaining interest; the mathematical equipment—its use and care (includes slide rule, blackboards, protractors and rulers, filing cases, homemade instruments—a transit made from erector set pieces is illustrated); mathematical library.

ART, ARCHITECTURE, AND DYNAMIC SYMMETRY

A Rhythmic Approach to Mathematics. Edith L. Somervell. Reprints available from Miss L. E. Christman, 1217 Elmdale Avenue, Chicago, 1906. 67 pages.

Introduces student to various geometric curves by means of sewing cards; e. g., the parabola as the pursuit curve of dog chasing a rabbit, the spiral of Archimedes, and many interesting designs.

Dynamarhythmic Design. Edward B. Edwards. New York: D. Appleton-Century Company, 1938. 122 pages.

Dynamic Symmetry—The Greek Vase. Jay Hambidge. New Haven: Yale University Press, 1920. 161 pages.

Analyses of ancient Greek vases based on the properties of ratios in rectangles.

Esthétique des Proportions dans la Nature et dans les Arts. Matila C. Ghyka. Paris: Gallimard, 1927. 452 pages.

Form in nature and art as related to mathematics. A valuable collection of material.

Nature's Harmonic Unity. Samuel Colman and C. Arthur Coan. New York: G. P. Putnam's Sons, 1912. 327 pages.

On Growth and Form. D'Arcy W. Thompson. Second Edition. Cambridge University Press, 1942. 1116 pages.

One of the best treatments of applications of mathematics in nature, including theory, discussion, and illustrations. Treats such topics as growth, soap films, capillarity, hexagonal symmetry, bee's cell, snow crystals, radiolarians, spider's web, spiral of Archimedes, nautilus shell, phyllotaxis.

Perspective Made Easy. Ernest R. Norling. New York: Macmillan Company, 1939. 203 pages.

An elementary, step-by-step introduction to the subject.

Proportional Form. Samuel Colman and C. Arthur Coan. New York: G. P. Putnam's Sons, 1920. 265 pages.

Deals with a mathematical analysis of art and nature. Studies are made of snow crystals, leaves, flowers, crystals, pine cones, shells, diatoms, butterflies and birds, fish and animals, the human figure, art and architecture.

Rhythmic Form in Art. Irma A. Richter. London: John Lane, 1932. 127 pages.

The principles of composition as found in the works of the great masters based on laws of perspective and dynamic symmetry.

The Elements of Dynamic Symmetry. Jay Hambidge. New York: Brentano's, 1926. 140 pages.

The logarithmic spiral is shown to give the approach to this form of design. Theory discussed in a series of twenty-one lessons.

The Frozen Fountain. Claude Bragdon. New York: Alfred A. Knopf, 1932. 125 pages.

Theory of ornament and architecture related to the five regular solids and regular polyhedrons of the fourth dimension.

PAPER FOLDING, SOAP FILMS, AND OTHER EXPERIMENTS

Fun with Paper. Joseph Leeming. New York: Stokes, 1939. 152 pages.

Includes the pentagon, Chinese tangrams, and various dissection puzzles.

Geometric Exercises in Paper Folding. T. S. Row. LaSalle, Ill.: Open Court Publishing Company, 1941. 148 pages.

How to form the square, equilateral triangle, pentagon, hexagon, octagon, nonagon, decagon and dodecagon, and pentadecagon. Also the conic sections.

Houdini's Paper Magic. Houdini. New York: E. P. Dutton and Company, 1922. 206 pages.

Paper folding and paper puzzles form two interesting chapters.

Soap Bubbles. C. V. Boys. London: Society for Promoting Christian Knowledge, 1902. Available through Macmillan Company, New York. 178 pages.

Many interesting experiments with soap solution resulting in minimal surfaces of interest to every high school student.

Spinning Tops. J. Perry. London: The Sheldon Press, 1929. Available through Macmillan Company, New York. 155 pages.

Theory and experiment with the gyroscope.

String Figures—An Amusement for Everybody. W. W. Rouse Ball. Cambridge: Heffer and Sons, 1921. 69 pages.

String designs, many originating with primitive people.

RECREATIONS AND AMUSEMENTS

Amusements in Mathematics. Henry E. Dudeney. New York: Thomas Nelson and Sons, 1940. 258 pages.

Reprinted as part of Nelson Black-Out Library.

Boy's Own Arithmetic. R. Weeks. New York: E. P. Dutton and Company, 1931. 188 pages.

Brain Teasers. P. J. Rulon. Boston: L. C. Page and Company, 1932.

Brain Tests. Walter B. Gibson. Boston: L. C. Page and Company, 1930. 224 pages, including answers.

Brush Up Your Wits. Hubert Phillips. Philadelphia: David McKay Company, 1936. 116 pages.

Caliban's Problem Book. Hubert Phillips. London: T. de la Rue, 1933. 340 pages.

Canterbury Puzzles. Henry E. Dudeney. New York: Thomas Nelson and Sons, 1940. 255 pages.

Diversions and Pastimes. R. M. Abraham. New York: E. P. Dutton and Company, 1935. 153 pages.

Games with paper, including dissection, paper compasses, five-pointed star; plans for crystal models, including the five regular polyhedrons; knots and string games.

Easy Number Tricks. Joseph C. Brown. Pelham, N. Y.: The author. 53 pages.

Fun with Figures. A. Frederick Collins. New York: D. Appleton-Century Company, 1928. 253 pages.

Mathemagic. R. V. Heath. New York: Simon and Schuster, 1933. 138 pages.

Mathematical Clubs and Recreations. S. I. Jones. Nashville, Tenn.: S. I. Jones Company, 1940. 250 pages.

Mathematical Nuts. S. I. Jones. Nashville, Tenn.: S. I. Jones Company, 1936. 340 pages.

Mathematical Recreations and Essays. W. W. Rouse Ball and H. S. M. Coxeter. Eleventh Edition. London: Macmillan and Company, 1939. 418 pages.

Geometrical dissections; tesselation, color cubes, tangrams; polyhedra, including the kaleidoscope; three classical geometrical problems; calculating prodigies. Excellent references included in all chapters.

Mathematical Wrinkles. S. I. Jones. Nashville, Tenn.: S. I. Jones Company, 1930. 361 pages.

Modern Puzzles. Henry E. Dudeney. New York: F. A. Stokes, 1926. 190 pages.

Numbergrams. N. Sparhawk. Boston: Van Press, 1932.

Puzzles and Curious Problems. Henry E. Dudeney. New York: Thomas Nelson and Sons, 1940. 195 pages.

Puzzle Papers in Arithmetic. F. C. Boon. London: G. Bell and Sons, 1937. 64 pages, including answers.

Recreations in Mathematics. H. E. Licks. New York: D. Van Nostrand Company, 1916. 155 pages.

The Book of Puzzles. A. Frederick Collins. New York: D. Appleton and Company, 1927. 190 pages.

Tangrams and the loculus of Archimedes, a few geometrical dissections.

There Is Fun in Geometry. Louis Kasper. New York: Fortuny's, 1937. 135 pages.

Winter Nights Entertainments. R. M. Abraham. New York: E. P. Dutton and Company, 1933. 186 pages.

Paper folding; string tricks and figures; knots, bends, and splices.

NUMBERS

New Numbers—How Acceptance of a Duodecimal Base Would Simplify Mathematics. F. Emerson Andrews. New York: Harcourt, Brace and Company, 1935. 168 pages.

Numbers and Numerals. David Eugene Smith and Jekuthiel Ginsburg. New York: Bureau of Publications, Teachers College, Columbia University, 1937. 56 pages.

Roman, Chinese, and Japanese abacus and the development of numbers.

Number Stories of Long Ago. David Eugene Smith. Boston: Ginn and Company, 1919. 150 pages.

Abacus, calculating machines, slide rules, and fun with numbers.

Odd Numbers, or Arithmetic Revisited. Herbert McKay. Cambridge University Press, 1940. Available through Macmillan Company, New York. 215 pages.

Rapid Arithmetic. T. O'Connor Sloane. New York: D. Van Nostrand Company, 1922. 198 pages.

Speed and Fun with Figures. T. O. Sloane, H. E. Licks, James E. Thompson. New York: D. Van Nostrand Company, 1939. 602 pages.

The Wonderful Wonders of One—Two—Three. David Eugene Smith. Fulton, N. Y.: McFarlane, Warde, McFarlane, 1937. 47 pages.

SOURCE BOOKS FOR FURTHER STUDY, AND MISCELLANEOUS

A Companion to Elementary School Mathematics. F. C. Boon. London: Longmans, Green and Company, 1924. 302 pages.

Mechanical construction of curves, including linkages; Pythagoras' theorem; symmetry; continuity; paradoxes and fallacies.

Algebra, an Interesting Language. E. R. Breslich. New York: Newson and Company, 1939. 70 pages.

Illustrates and discusses applications of algebra and arithmetic in life around us.

A Scrapbook of Elementary Mathematics. William F. White. LaSalle, Illinois: Open Court Publishing Company, 1910. 248 pages. (Out of print.)

Napier's rods and other mechanical aids to calculation, geometric puzzles, a home-made leveling device, "rope stretchers," the three famous problems of antiquity, linkages and straight line motion, paper folding, apparatus to illustrate line values of trigonometric functions. Extensive bibliographic index.

Curiosités Geometriques. E. Fourrey. Paris: Librairie Vuibert, 1938. 431 pages.

Pythagorean Theorem, dissection figures, surveying instruments.

Flatland—a Romance in Many Dimensions. "A Square." (E. A. Abbott) Rev. Ed. Boston: Little, Brown and Company, 1926. 155 pages.

Fundamental Mathematics. Duncan Harkin. New York: Prentice-Hall, Inc., 1941. 426 pages.

Mathematics and the Imagination. Edward Kasner and James Newman. New York: Simon and Schuster, 1940. 380 pages.

Mathematics for the Million. Lancelot Hogben. New York: W. W. Norton, Inc., 1937. 647 pages.

Mathematics, Its Magic and Mastery. Aaron Bakst. New York: D. Van Nostrand Company, 1941. 790 pages.

Abacus, Napier's rods; also modification of Napier's rods in the form of a calculating apparatus using ten sets of Napier's rods.

Procédés Originaux de Constructions Géométriques. E. Fourrey. Paris: Librairie Vuibert, 1924. 142 pages.

Constructions with compasses alone, and ruler alone. Paper folding.

The Pythagorean Proposition. Elisha S. Loomis. Second Edition. Berea, Ohio: Professor O. L. Dustheimer, 1940. 214 pages. $2.00.

Two hundred and fifty-six proofs of this famous theorem.

Trisection. Robert C. Yates. Baton Rouge, Louisiana: Franklin Press, 1941. $1.00.

A thorough treatment of the subject.

What Is Mathematics? Richard Courant and Herbert Robbins. New York: Oxford University Press, 1941. 521 pages.

Includes soap film experiments; an instrument for doubling the cube; linkages.

HISTORY OF MATHEMATICS

A Primer of the History of Mathematics. W. W. R. Ball. New York: Macmillan Company, 1922. 149 pages.

A Short History of Mathematics. Vera Sanford. Boston: Houghton Mifflin Company, 1930. 384 pages.

Many instruments and devices are discussed and illustrated.

History of Mathematics. David Eugene Smith. Volume 1, 1925, 570 pages; Volume 2, 1925, 703 pages. Boston: Ginn and Company.

Number, the Language of Science. Tobias Dantzig. Third Edition. New York: Macmillan Company, 1939. 320 pages.

The History of Arithmetic. Louis C. Karpinski. Chicago: Rand, McNally Company, 1925. 200 pages.

TABLES, BIBLIOGRAPHY OF AIDS, DICTIONARY

A Bibliography of Mathematical Education. William L. Schaaf. Forest Hills, N. Y.: Stevinus Press, 1941. 144 pages.

A classified index of periodical literature since 1920, containing over 4,000 references. Section on audio-visual aids includes mathematical models, exhibits, films and slides as well as general aids. Also references on mathematical plays and dialogues, clubs and recreations, general extra-class activities, use of historical material, graphs, slide rule, and logarithms.

Enriched Teaching of Mathematics in the Junior and Senior High School. A source book of illustrative and supplementary materials for teachers of mathematics. Maxie N. Woodring and Vera Sanford. Revised Edition. New York: Bureau of Publications, Teachers College, Columbia University, 1938. 133 pages.

Materials for units in arithmetic, algebra, geometry, and trigonometry, as well as tests and workbooks, are listed. Also references to articles, books, and names of equipment dealers for lantern slides, motion pictures, slide rules, surveying instruments, pictures, bulletin boards and exhibits, assemblies, clubs, plays, contests, and recreations.

Handbook of Mathematical Tables and Formulas. R. S. Burington. Second Edition. Sandusky, Ohio: Handbook Publishers, Inc., 1940. 275 pages.

Lefax. A pocket-size system of loose-leaf data sheets and blank forms. Philadelphia: Lefax, Inc.

Loose-leaf sheets listing formulas for various branches of mathematics, engineering, and business, as well as all forms of tables. Sheets are sold individually or in groups and may be inserted in notebook.

Mathematical Tables from Handbook of Chemistry and Physics. C. D. Hodgman. Revised Ed. Cleveland: Chemical Rubber Publishing Company, 1941. 178 pages.

Formulas for algebra, geometry, trigonometry, and higher mathematics. Tables for trigonometric functions, logarithms, radians, compound interest and annuities, and higher mathematics.

Mathematics Dictionary. Glenn James and Robert C. James. Van Nuys, Calif.: The Digest Press, 1942. 259 pages.

Basic words and topic phrases used in arithmetic, elementary algebra, plane and solid geometry, trigonometry, and junior college mathematics. Appendix gives formulas and tables.

Sources of Visual Aids for Instructional Use in Schools. Pamphlet No. 80, Revised. Federal Security Agency, U. S. Office of Education. Washington, D. C.: Superintendent of Documents, 1941. 91 pages.

Lists sources of visual aids for all departments under the headings charts and graphs, exhibits, filmstrips, slide films, lantern slides, maps and globes, motion pictures, objects, specimens and models, pictures and photographs, posters and cartoons, stereographs and other three-dimensional pictures. Also dealers in such equipment as cameras, lantern slides and filmstrips, filmstrip and stillfilm projectors and attachments, lantern projectors, micro-projectors, motion-picture projectors, opaque projectors.

The Audio-Visual Handbook. Ellsworth C. Dent. Fourth Edition. Chicago: Society for Visual Education, Inc., 1942. 227 pages.

Discusses all forms of visual and audio aids. Classroom experiments and blackboard demonstrations, plays and pageants, models, school museum, graphs, maps and globes, photographs, opaque projection of pictures, stereograph and stereoscope, various types of slides—how they are made, how they are used—equipment, three-dimensional projection, silent and sound pictures. Sources of information, materials, and equipment are also given under many of the headings listed above, as well as sources for pamphlets and bulletins, and organizations from whom information may be obtained.

Visual Aids in Education. A Syllabus for Use in Classes in Schools and Colleges of Education. G. W. Leman. Published by State Teachers College, Paterson, N. J. 71 pages.

Handmade lantern slides, the standard slide projector and its uses, the photographic lantern slide, the blackboard and bulletin board, flat pictures, other graphic aids, the stereograph and stereoscope, opaque projection, the filmslide and the stillfilm, the motion picture.

FILMS AND FILMSTRIPS

FILMS

The following films, unless otherwise noted, are available in sound only and are 16 millimeters in width. Most of them are one reel (about 400 feet) and require about fifteen minutes for projection. Some may be purchased from the producer or rented from film lending agencies or rental libraries. Rental prices vary according to the length of film and the type and location of the lending agency.[1] Prices should be checked before ordering. Teachers are advised to write to distributors or libraries for further information, or to consult the director of visual education in their own city school system, who will have catalogs on file. For titles of new films, they should consult recent issues of *Educational Film Catalog,* published several times each year by the H. W. Wilson Company, 950 University Ave., New York, N. Y.

Origin of Mathematics. Deals with the history of numbers and geometry. Produced in 1941 by H. W. Allers, 960 So. Oxford Ave., Los Angeles, Calif.

Geometry in Action. Suitable for use in junior high school or as an introduction to the course in plane geometry. Produced in [illegible]940 by Bald Eagle Film Productions, 104 Howe St., Annex, New Haven, Conn.

Rectilinear Coordinates. An introduction to plane and solid analytic geometry. Produced in 1939 by McCrory Studios, 130 W. 46th St., New York, N. Y.

The Slide Rule. Explains the "C" and "D" scales and their use in multiplication and division. 850 feet, $30.67. Filmstrip also available, $1.00. Free manual. Castle Films, Inc., 30 Rockefeller Plaza, New York, N. Y.

Frequency Curves. May be used in an introductory course in statistics at the high school or college level. Silent only, ½ reel. Produced by Eastman Kodak Company, Teaching Films Division, Rochester, N. Y.

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[1] See also E. H. C. Hildebrandt, *Mathematics Films,* Mathematics Teacher, Vol. 34, pp. 26–34, January, 1941.

Einstein's Theory of Relativity. Contains some good material on the subject although photographic methods have improved considerably since the film was produced in 1924. Silent, 2 reels. Bell and Howell Company, 1801–1815 Larchmont Ave., Chicago, Ill.

Stereoscopic Mapping from the Air. Illustrates processes for making topographic maps from aerial photographs. Produced prior to 1940. Bell and Howell Company, 1801–1815 Larchmont Ave., Chicago, Ill.

Precisely So. Shows the history and development of standards of measure and the need for accuracy in the production of engines. Produced in 1937 by Chevrolet Motor Division of General Motors Corporation, Detroit, Mich.

Measurement with Light Waves. Shows how light waves are used in obtaining measurements to millionths of an inch. Also demonstrates gage block inspection methods. 550 feet, $22.73. Filmstrip available, $1.00. Free manual. Castle Films, Inc., 30 Rockefeller Plaza, New York, N. Y.

Fundamental Concepts of Measure. The following five films were produced under the direction of the U. S. Office of Education to help in training workers for war industries. They can be used effectively in mathematics classes to emphasize the fundamental concepts of measure. They may be purchased from Castle Films, Inc., 30 Rockefeller Plaza, New York, N. Y. An additional filmstrip for $1.00 and a free manual are also available to supplement each film, or these may be secured separately.

The Steel Rule. 500 feet, $21.49.

The Micrometer. 525 feet, $22.73.

Fixed Gages. 600 feet, $23.98.

Vernier Scale. 675 feet, $25.85.

Height Gages and Standard Indicators. 425 feet, $18.59.

Principles of Aerial Navigation. Several films have been produced for teaching the principles of aerial navigation in the Army and Navy air schools. These are also available from Castle Films, Inc., New York, N. Y. Filmstrips were not prepared for these films.

Celestial Navigation—Part I: Introduction and Location of Celestial Points. 650 feet, $16.98.

Celestial Navigation—Part II: Principles of Position Finding on the Earth. 525 feet, $14.10.

Aerial Navigation—Part I: Maps and Compasses. 500 feet, $13.52.

Celestial Navigation—The Earth. 600 feet, $15.83.

Celestial Navigation—Charts. 675 feet, $17.56.

Nautical Astronomy. 850 feet, $22.03.

FILMSTRIPS

Filmstrips, also called slidefilms or picturols, are available from a number of producers and distributors. They require the use of a 35-millimeter filmstrip projector. They have been used extensively, along with motion pictures, as basic training aids by the Army, Navy, Marine Corps, and Coast Guard, as well as by industries and government agencies.

ALGEBRA

The Origin of Algebra, Roll I

Basic Definition of Algebra, Roll II

Each film (no manual) $2.00, from Society for Visual Education, Inc., 100 E. Ohio St., Chicago, Ill.

GEOMETRY

Introduction to Plane Geometry
Rectilinear Figures—Triangles
Rectilinear Figures—Triangles and Parallel Lines
Rectilinear Figures—Angles and Triangles
Rectilinear Figures—Quadrilaterals and Polygons
Rectilinear Figures—Polygons and Locus of a Point
The Circle—Chords and Arcs
The Circle—Tangents and Secants
The Circle—Measurement of Angles

Each film (no manual) $2.00, from Society for Visual Education, Inc., 100 E. Ohio St., Chicago, Ill.

LIGHT ON MATHEMATICS

Kit I: Arithmetic
  Five Keys to Mathematics
  Addition and Subtraction
  Multiplication and Division
  Fractions, Decimals, and Percentage
  Addition and Subtraction of Fractions
  Multiplication and Division of Fractions
  Square and Cube Root
Kit II: Geometry and Trigonometry
  Addition and Subtraction in Geometry
  Multiplication and Division in Geometry
  Angular Measurement
  Constructions
  Scales and Models
  Vectors
  Trigonometry
Kit III: Algebra
  Positive and Negative Numbers
  Ratio and Proportion
  Exponents and Logarithms
Kit IV: Graphs and Analytic Geometry
  Graph Uses
  Plotting Graphs
  Analytic Geometry

Entire set of four kits may be purchased together or the films may be obtained separately at $4.00 each. The Jam Handy Organization, 2900 E. Grand Blvd., Detroit, Mich.

MATHEMATICAL INSTRUMENTS

Measurement Instruments—In the Laboratory
Measurement Instruments—Outside the Laboratory
Calculation Instruments: Graphical Methods, Slide Rules, and Solving Equations

Calculation Instruments: Business Machines—Part I: Early Methods
Each film (with manual) $2.00, from Society for Visual Education, Inc., Chicago, Ill.
The Slide Rule
One film (with manual) $1.00, from Castle Films, Inc., 30 Rockefeller Plaza, New York, N. Y.

GEOMETRIC DRAWING AND BLUE PRINT READING
Geometric Construction, Parts 1 and 2
T Square and Triangles, Parts 1 and 2
Each film (with manual) $1.00, from The Jam Handy Organization, Detroit, Mich.
Blue Print Reading
Film (with manual) $1.00, from Castle Films, Inc., New York, N. Y.

SCIENCE
Optical Illusions
Timekeepers Through the Ages (Clocks)
Mechanics I (Levers)
Mechanics II (Inclined Plane)
Mechanics III (Pulleys—Wheel and Axle)
Each film (no manual) $2.00, from Society for Visual Education, Inc., Chicago, Ill.

MEASURE
The Steel Rule
The Micrometer
Fixed Gages
Vernier Scale
Height Gages and Standard Indicators
Each film (with manual) $1.00, from Castle Films, Inc., New York, N. Y.
Measurements and Measuring, Parts 1 and 2
Each film (with manual) $4.00, from The Jam Handy Organization, Detroit, Mich.

NAVIGATION
Aerial Navigation: Dead Reckoning Problems—Part I: Radius of Action, Returning to Same Base
Aerial Navigation: Dead Reckoning Problems—Part II: Radius of Action, Returning to Alternate or Moving Base
Each film (no manual) $0.72, from Castle Films, Inc., New York, N. Y.

OTHER MATERIALS

Further suggestions on free and inexpensive materials, projection equipment, etc., will appear from time to time in the *Mathematics Teacher.* Suggestions are also offered in the publication, *Enriched Teaching of Mathematics in the Junior and Senior High School,* by M. Woodring and V. Sanford (Bureau of Publications, Teachers College, New York, N. Y.) and in a mimeographed list entitled "Mathematics—Visual and Teaching Aids" by L. Heimers and E. H. C. Hildebrandt (Visual Aids Service, Montclair State Teachers College, Montclair, N. J.).